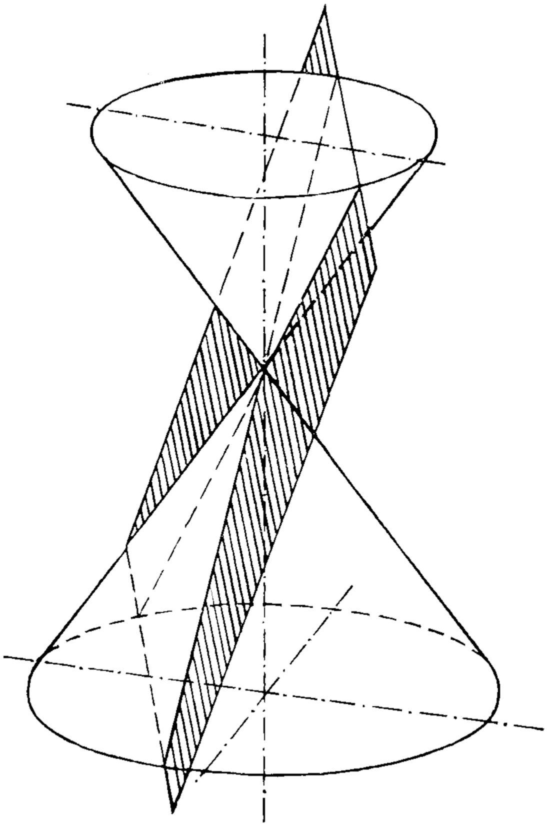
A COURSE IN DESCRIPTIVE GEOMETRY

V. O. GORDON and M. A. SEMENTSOV-OGIEVSKII



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A COURSE
IN DESCRIPTIVE GEOMETRY

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AND M. SEMENTSOV-OGIEVSKII, D. Sc.

This celebrated textbook was written as a course in descriptive geometry for students of machine building and mechanical engineering institutes.
It contains sections on systems of orthogonal projections and axonometric projections. It also includes a systematic account of topics like formation of projections, point and straight line, planes and their mutual positions, formation of a drawing by varying projection planes and rotation, construction of polygons and their intersection by a plane and a straight line, intersection of one polygonal surface by another, curved lines and curved surfaces, intersection of curved surfaces by a plane and a straight line, intersection of one curved surface by another, involution of curved surfaces.
Each chapter contains a large number of test questions to make the subject matter more readily comprehensible.
The book also contains a short account of the development of descriptive geometry since XIX century as well as information about some scientists who have made significant contributions in the field of descriptive geometry both as a science and as a subject of study.

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ИЗДАТЕЛЬСТВО · МИР ·

КУРС НАЧЕРТАТЕЛЬНОЙ ГЕОМЕТРИИ

В. О. ГОРДОН,
М. А. СЕМЕНЦОВ-ОГИЕВСКИЙ

ИЗДАТЕЛЬСТВО «НАУКА»
МОСКВА

A COURSE IN DESCRIPTIVE GEOMETRY

V. O. GORDON
and M. A. SEMENTSOV-OGIEVSKII

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CONTENTS

INTRODUCTION

Descriptive geometry is one of the fundamental disciplines making up an engineering education.

It is concerned with setting forth and justifying methods of constructing representations of three-dimensional forms in the plane, as well as methods of solving geometrical problems on the basis of given representations of these forms. As is known, three-dimensional forms can be represented not only in the plane, but on some other surface, for instance, a cylinder or sphere. The latter cases are studied in special branches of descriptive geometry.

The representations constructed according to the rules of descriptive geometry enable us to visualize the shape of objects and their relative positions in space, to determine their dimensions, and to study their geometrical properties.

Descriptive geometry develops the student's three-dimensional imagination by making frequent appeals to it.

Finally, descriptive geometry provides a number of practical means for engineering drawings, ensuring their clarity and accuracy, and, hence, the possibility of manufacturing the represented objects.

The rules for constructing representations, set forth in descriptive geometry are based on the method of projections.

It is standard practice to begin studying the method of projection with the construction of the projections of the *point*, since the construction of the projections of any three-dimensional form involves considering a number of points belonging to this form.

CHAPTER 1

THE METHOD OF PROJECTING

Sec. 1. Central Projections

To obtain *central projections* we must take a plane (the *plane of projection*) and a fixed point not in the plane (the *centre of projection*). The *method of central projection* is illustrated in Fig. 1 showing the plane P and the point S. Taking a point A and drawing through S and A a straight line, we intersect the plane P at point a_p. We then proceed in the same way with the points B and C. The points a_p, b_p, c_p are *central projections* of the points A, B, C on the plane P: they are obtained as the intersections of the *projecting lines* (or rays called the projectors) SA, SB, SC with the plane of projection*.

If for a certain point D (Fig. 1) the projector turns out to be parallel to the plane of projection, then we conventionally consider that they intersect, but at a point at infinity. The point D also has a projection which is an infinitely distant point (d_∞).

Leaving the position of the plane P unchanged and taking a new centre S_1 (Fig. 2), we obtain a new projection of the point A (point a_{p1}). If the centre S_2 is taken on the same projector SA, then the projection a_p remains unchanged.

Hence, given the plane and the centre of projection, we can construct the projection of a point (Fig. 1), but having the projection of a point (for instance, a_p) it is impossible to determine the position of the point A in space, since any point on the projector SA is projected into one and the same point. Obviously, for obtaining the unique solution some additional conditions are required.

The projection of a line can be constructed by projecting a number of its points (Fig. 3), all the projectors generating a conical surface** or being

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*The centre of projection is also called the *pole of projection*, and central projection is termed '*polar projection*'.

**That is why central projections are also called *conical*.

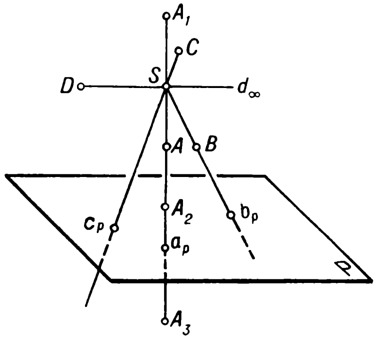


Fig. 1

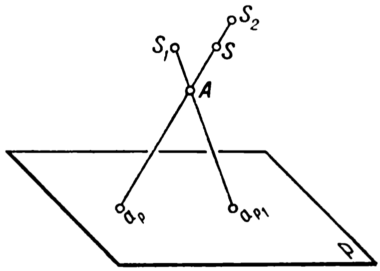


Fig. 2

located in one plane (for instance, when projecting a straight line not passing through the centre of projection, or a polygonal line and a curve all points of which lie in a plane coinciding with the projecting plane).

Obviously, the projection of a line is obtained as the intersection of the projecting surface with the plane of projection (Fig. 3). But, as is shown in Fig. 4, the projection of a line does not determine the line being projected, since the projecting surface may contain a number of lines which are projected on the plane of projection into one and the same line.

From the projecting of points and lines we may pass over to projecting a surface and a solid.

Sec. 2. Parallel Projections

Let us now consider the *method of parallel projection.*

When the centre of projection is a point at infinity, all the projections are parallel. They are drawn in the direction indicated by an arrow (see Fig. 5). *The projections constructed in such a way are called parallel.*

Thus, parallel projection may be considered as a particular case of central projection.

Hence, *the parallel projection of a point is defined as the point of intersection of a projector drawn parallel to a given direction with the plane of projection.*

To obtain a parallel projection of a line it is sufficient to construct projections of a number of its points and to draw through them a line (Fig. 6).

In this case all the projectors form a cylindrical surface, therefore parallel projections are also called *cylindrical.*

In parallel projections, the same as in central projections in general:

(1) for a straight line the projecting surface in the general case is a plane, and therefore a straight line is, in general, projected into a straight line;

(2) any point and line in space have its unique projection;

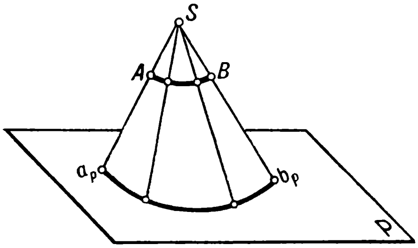


Fig. 3

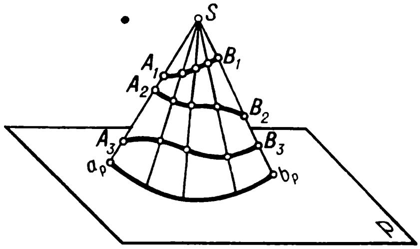


Fig. 4

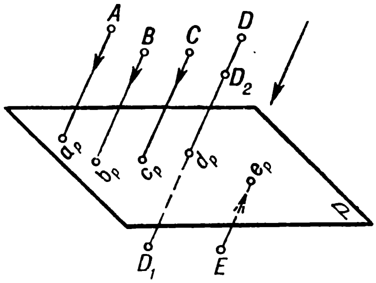


Fig. 5

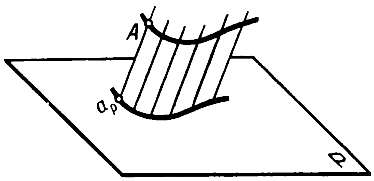


Fig. 6

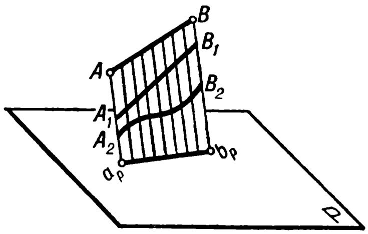


Fig. 7

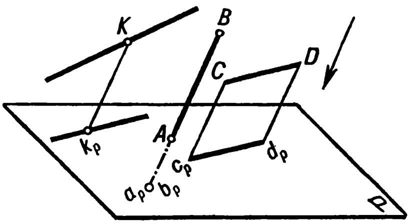


Fig. 8

(3) each point on the projection plane may be the projection of a set of points if they are situated on a straight projector (the point d_p in Fig. 5 is the projection of points D, D_1, D_2);

(4) any line on the projection plane may turn out to be the projection of a set of lines if they are contained in a common projecting plane (Fig. 7: the line segment a_pb_p serves as the projection of line segments AB and A_1B_1 of straight lines and the segment A_2B_2 of a plane curve); obviously, to obtain the unique solution in this case, some additional conditions are required;

(5) to construct the projection of a straight line it is sufficient to project two of its points and to draw a straight line through the obtained projections of these points;

(6) if a point belongs to a straight line, then the projection of the point belongs to the projection of this line (point K in Fig. 8 belongs to a straight line, and the projection k_p belongs to the projection of this line).

In addition to the above listed properties the following is valid for parallel projections:

(7) if a straight line is parallel to the direction of projecting (as, for instance AB in Fig. 8), then the projection of the line (and any of its segments) is a point (a_p, or b_p);

(8) a segment of a straight line parallel to the plane of projection is projected on this plane true length (Fig. 8: CD is equal to c_pd_p as segments of parallel lines between parallel lines).

Later on we shall consider some more properties of parallel projections showing what relationships inherent in objects under considerations are retained in the projections of these objects.

Applying the methods of parallel projection of a point and a line, it is possible to construct parallel constructions of a surface and a solid.

Parallel projections are subdivided into oblique and orthogonal projections. In the first case the direction of projecting forms with the plane of projection an angle not equal to 90°, whereas in the second case the projectors are perpendicular to the plane of projection.

When considering parallel projections the viewer should be imagined as located at an infinite distance from the image. But in reality objects and their images are viewed from a finite distance, and the rays entering viewer's eye form a conical, but not a cylindrical, surface. Hence, a more natural picture is obtained (provided certain conditions are observed) using a central projection, but not a parallel one. Therefore, when it is required to get a representation producing the same visual impression as the object itself, we usually resort to perspective projections which are based on central projecting.

But despite the above mentioned conditionality parallel projecting is widely applied. This is explained by the properties of parallel projections as well as by a comparatively greater simplicity of the constructions involved.

Sec. 3. Monge's Method

Information and methods of construction required for representing space forms in the plane have accumulated gradually since ancient times. During long period of time plane representations were accomplished mostly in a visualized manner. With the development of engineering paramount importance was acquired by the need of developing a method which would ensure accuracy and easiness in measuring graphical representations, i.e. ensure the possibility to locate each point of the representation relative to other points or planes and to determine the dimensions of line segments and figures by simple methods.

The accumulated rules and methods for constructing such representations were systemized and further developed by the great French mathematician G. Monge, the inventor of descriptive geometry, in his work "Essais sur les Géométrie déscriptive" issued in 1779.

Gaspard Monge (1746-1818) is known in history as a great French mathematician, engineer, a public man and a stateman during the period of Revolution of 1789-94 and the rule of Napoleon.

In 1768 Monge became professor of mathematics and in 1771 professor of physics at Mézières; in 1780 he was appointed to a chair of hydrolics at the Lyceum in Paris (held by him together with his appointments at Mézières) and was received as a member of the Académie.

Monge wrote various mathematical and physical papers.

He took an active part in the measures for the establishment of the normal school and of the well-known École Polytechnique (Polytechnic school) and was at each of them professor for descriptive geometry.

Being one of the ministers (Minister of Marine) in the revolutionary government of France, Monge did much for its defence against foreign invaders, as well as for the victory of the revolutionary troops.

For a long time Monge had no possibility to publish his work containing the description of the method elaborated by him. It was considered so valuable that it long was guarded as military secret. Only at the very end of the 18th century the prohibition to publish his book was rescinded by the French government, and in 1799 Monge issued the mentioned work in which he gave a comprehensive description of his method.

On the fall of Napoleon he was deprived, as a Bonapartist, of all his honours and excluded from the list of members of the reconstructed bodies. He was forced to hide, and ends his life in poverty.

The method of parallel projection (with orthogonal projections on two mutually perpendicular planes of projection) invented by Monge was and remains the principal method applied for making engineering drawings, since it ensures obviousness, accuracy, and easiness in measuring representations of various objects in the plane.

As a result of his researches, Monge arrived at that general method of the application of geometry to the arts of construction that later became known as descriptive geometry.

The present course deals preferably with orthogonal projections, which are a particular case of parallel oblique projections. If the latter are used, it will be mentioned each time.

QUESTIONS TO CHAPTER 1

1. How do we construct the central projection of a point?
2. In what case does the central projection of a straight line represent a point?
3. What does the method of parallel projection consist in?
4. How is the parallel projection of a straight line constructed?
5. May the parallel projection of a straight line represent a point?
6. If a point belongs to a given straight line, then what are their relative positions?
7. In what case of parallel projection is a segment of a straight line projected true length?
8. What is the Monge's method?
9. How is the word aorthogonal' deciphered?

CHAPTER 2

THE POINT AND THE STRAIGHT LINE

Sec. 4. A Point in the System V, H

As it was mentioned in Sec. 2, the projection of a point does not define the position of the latter in space, so to fully define this position some additional conditions are required. For instance, we are given the orthogonal projection of a point on the horizontal plane of projection and its distance from this plane is indicated by an elevation. The plane of projection is then taken for "the plane of zero level", and the elevation is said to be *positive* if a point in space is above the plane of zero level, and *negative* if a point is below this plane.

This is the essence of the *method of projections with elevation.* We are not going to study this method in further detail.

In our book we shall define the positions of points in space by their orthogonal projections on two and more planes of projections.

Figure 9 depicts two mutually perpendicular planes. Let us take them for the planes of projections. One of them denoted by the capital letter H is horizontal, the other denoted by the capital letter V is vertical. The latter plane is called the *vertical plane of projection,* the plane H being the *horizontal plane of projection.* The projection planes V and H form the system V, H.

The line of intersection of the projection planes is called the *axis of projection.* It divides either of the planes V and H into half-planes. This axis will be designated x, or in the form of a common fraction V/H. The projection planes divide the space into four dihedral angles or quadrants, the first one being the quadrant whose faces are designated V and H in Fig. 9.

The construction of the projections of a point A in the system V, H is illustrated in Fig. 10. Drawing from A perpendiculars to V and H, we obtain the projections of the point A: the *vertical* projection designated a', and the *horizontal* projection designated a.

The projecting rays (or projectors) respectively perpendicular to the planes V and H define a plane which is perpendicular to the planes and axis of

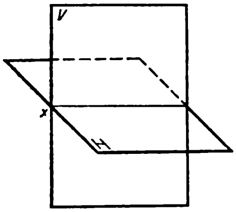


Fig. 9

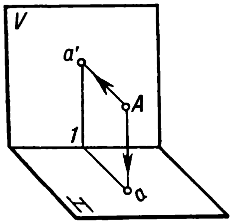


Fig. 10

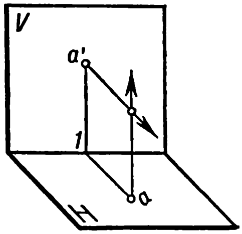


Fig. 11

projection. This plane, when intersecting V and H, forms two mutually perpendicular straight lines $a'1$ and $a1$ which intersect at point 1 on the axis of projection. Consequently, *the projections of a point are always situated on straight lines perpendicular to the axis of projection and intersecting this axis in one and the same point.*

If the projections a' and a of a point A are given (Fig. 11), then, drawing perpendiculars—through a' to the plane V and through a to the plane H—we get a definite point which is the point of intersection of these perpendiculars. Hence, *two projections of a point entirely define its position in space relative to a given system of projection planes.*

Rotating the plane H about the axis of projection through an angle of 90° (as is shown in Fig. 12), we shall get a single plane, i.e. the plane of the drawing; the projections a' and a will be located on a single perpendicular to the axis of projection (Fig. 13). Let us agree to call the straight lines joining different projections of a point *lines of recall.*

As a result of bringing to coincidence the planes V and H, we obtain a projection drawing termed the orthographic representation (or Monge's representation). This is a drawing in the system V, H (or in the system of two orthogonal projections).

Passing over to such a representation, we lose the three-dimensional picture of arrangement of the planes of projection and a point. But, as we shall see later on, an orthographic representation ensures accuracy and convenience in measuring the represented elements along with simplicity of constructions. To get a three-dimensional picture from an orthographic representation one should possess a power of imagination. For instance, given Fig. 13, we have to imagine the picture represented in Fig. 10.

Since in the presence of the axis of projection the position of the point A relative to the planes of projections V and H is determined, the line segment $a'1$ represents the distance of the point A from the projection plane H, and the line segment $a1$, the distance of the point A from the plane V. We can also determine the distance of the point A from the axis of projection. It is represented by the hypotenuse of the right triangle constructed on $a'1$ and $a1$ as its legs (Fig. 14): laying off on the orthographic representation the line segment $a'\bar{A}$ equal to $a1$ and perpendicular to $a'1$, we get the true length of the hypotenuse $\bar{A}1$ representing the required distance.

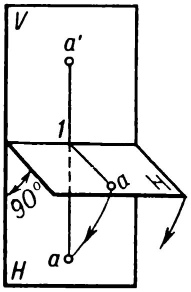


Fig. 12

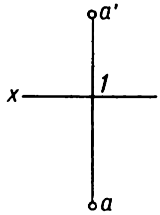


Fig. 13

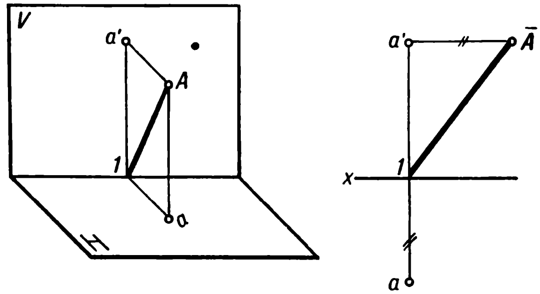


Fig. 14

We conclude with the following note: the *line of recall* connecting different projections of a point should be necessarily drawn, since the presence of this line makes it possible to determine the required position of a point.

Let us agree here to call Monge's representation, as also *the drawings based on Monge's method* (see Sec. 3), *by one word—the drawing, and understand it only in this sense.* If used otherwise, the word "drawing" will be preceded by a corresponding attribute, say, a perspective drawing, an axonometric drawing, and so on.

Sec. 5. A Point in the System *V, H, W*

In a number of constructions and when solving problems, it becomes necessary to introduce other planes of projection in the system V, H. It is known that in mechanical engineering the drawings of machines and their parts contain not two but more representations (views).

Let us consider the introduction of one more plane of projection into the system V, H (Fig. 15). This plane denoted by the capital letter W is perpendicular both to V, and to H, and is called the *profile plane of projection.* Like the plane V, the plane W occupies the vertical position. In addition to the x-axis of projection, we obtain two more axes (the z- and y-axes) which are perpendicular to the x-axis. The point of intersection of all the three planes of projections is designated by the capital letter O. Since the x-axis is perpendicular to the plane W, the y-axis is perpendicular to the plane V, and the z-axis is perpendicular to the plane H, in the point O there coincide the projections of the x-axis on the plane W, of the y-axis on the plane V and of the z-axis on the plane H.

Figure 15 shows how the planes H, V, and W are brought into coincidence with the plane of the drawing. For the y-axis two positions are given (Fig. 17).

The pictorial representation in Fig. 16 and the drawing of Fig. 18 contain the horizontal, vertical, and profile projections of a point A.

The horizontal and vertical projections (a and a') are situated on one perpendicular to the x-axis, i.e. *on the line of recall* $a'a$, the vertical and

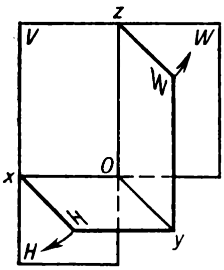


Fig. 15

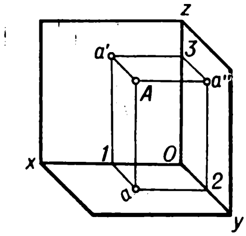


Fig. 16

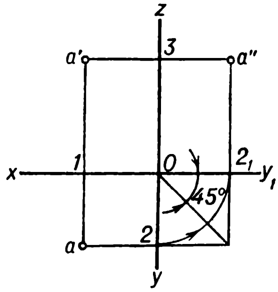


Fig. 1 .

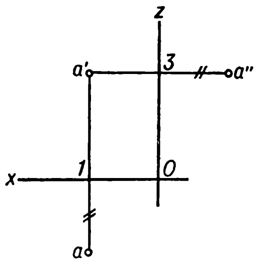


Fig. 18

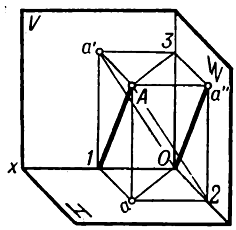


Fig. 19

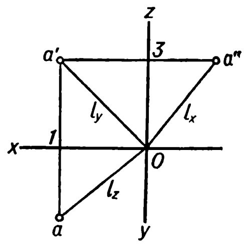


Fig. 20

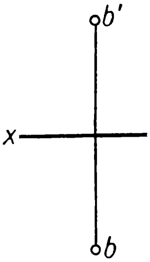


Fig. 21

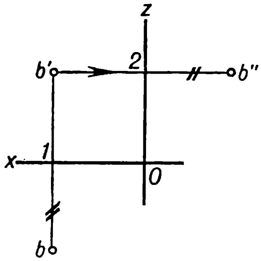


Fig. 22

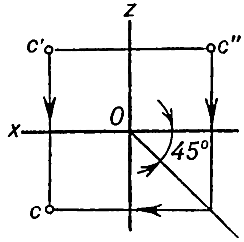


Fig. 23

profile projections (a' and a'') on one perpendicular to the z-axis, i.e. *on the line of recall $a'a''$.*

The construction of the profile projection from the vertical and horizontal projections is shown in Fig. 17. We can take advantage either of a circular arc described from the point O, or of the bisector of the angle yOy_1.

The distance of the point A from the plane H is measured on the drawing by the line segment $a'1$ or by the line segment $a''2_1$, the distance from the plane V by the line segment $a1$ or $a''3$, and its distance from the plane W by the line segment $a2$ or $a'3$. Therefore the projection a'' can also be constructed in the way shown in Fig. 18, i.e. by laying off on the line of recall joining the projections a' and a'' a line segment equal to $a1$ to the right of the z-axis. *This construction is preferable.*

The distance from the point A to the x-axis is measured in space by the line segment $A1$ (Fig. 19). But the line segment $A1$ is equal to the line segment $a''O$ (see Sec. 2, item 8). Therefore, for determining the distance from the point A to the x-axis on the drawing (Fig. 20) we have to take the line segment designated l_x.

Analogously, the distance from the point A to the y-axis is represented by the line segment l_y, and the distance from the point A to the z-axis by the line segment l_z (Fig. 20).

Thus, *the distances of a point from the projection planes and from the axes of projections can be measured directly as definite line segments on the drawing*, taking into account its scale.

Let us consider a few examples of construction of the third projection of a point, using the two given projections. Let a point B be given by its vertical and horizontal projections (Fig. 21). Introducing a z-axis (in Fig. 22 the distance $O1$ is arbitrary, if there are no particular conditions) and drawing through b' a line of recall perpendicular to the z-axis, we lay off on it to the right of this axis a line segment $b''2$ equal to $b1$.

Figure 23 shows how the projection c is constructed given the projections c' and c'' (the construction is indicated by arrows).

QUESTIONS TO SECS. 4-5

1. What is the "V, H system" and how are the projection planes V and H called?

2. What is the axis of projection?

3. How is the drawing of a point obtained in the V, H system?

4. What is the "system V, H, W" and how is the projection plane W called?

5. What is the line of recall?

6. How is it proved that a drawing containing two interconnected projections in the form of points represents a point?

7. How is the profile projection of a point constructed by its vertical and horizontal projections?

Sec. 6. Orthogonal Projections and a System of Rectangular Coordinates

The model representing the position of a point in the system V, H, W (shown in Fig. 16) is analogous to a model which can be constructed if the rectangular coordinates* of this point are known, i.e. the numbers corresponding to its distances from three mutually perpendicular planes which are called the *coordinate planes*. The straight lines along which the coordinate planes intersect are called the *coordinate axes*. The point of intersection of the coordinate axes is called the origin of coordinates, or simply the origin** and is designated by the capital letter O. For the coordinate axes we shall use the notation shown in Fig. 16.

Intersecting at right angles, the coordinate planes form eight trihedral angles, thus dividing space into eight parts called octants. Figure 16 represents one of the octants showing how the line segments defining the coordinates of a point A are obtained: perpendiculars are drawn from the point A to each of the coordinate planes. The first coordinate of the point A, called the *abscissa*, is expressed by the number obtained by comparing the line segment Aa'' (or an equal line segment $O1$ on the x-axis) with the line segment taken for the scale unit. In the same way the line segment Aa' (or an equal line segment $O2$ on the y-axis) will define the second coordinate of the point A, called the *ordinate;* the line segment Aa (or an equal line segment $O3$ on the z-axis) will define the third coordinate of the point A, called the *z-coordinate.*

In the literal notation of coordinates the abscissa is designated with a lower-case letter x, the ordinate with y and the z-coordinate with z.

The parallelepiped constructed in Fig. 16 is said to be a *coordinate parallelepiped* of a given point A. The construction of a point by its coordinates is reduced to constructing the three edges of the coordinate parallelepiped which form a three-segment polygonal line (see Fig. 24). We have to lay off in succession the line segments $O1$, $1a$, and aA, or $O2$, $a''2$, and $a''A$ and so on, i.e. point A can be obtained by six different combinations each of which must contain all the three coordinates.

For pictorial representation of the three-segment polygonal line we use in Fig. 24 the so-called *cabinet* projection*** in which the x- and z-axes are mutually perpendicular, the y-axis being the extension of the bisector of the angle xOz. In the cabinet projection the line segments laid off on the y-axis or parallel to it are shortened by half.

*Or the rectangular Cartesian coordinates. Cartesian system of coordinates may be rectangular or oblique; here we consider only a rectangular system of coordinates.

Rene Descartes (1596-1650), celebrated French philosopher and mathematician. Cartesius is the latinized version of his name.

**The initial letter of the Latin word 'origo', the beginning.

***The cabinet projection belongs to oblique projections (for more detail see Sec. 75).

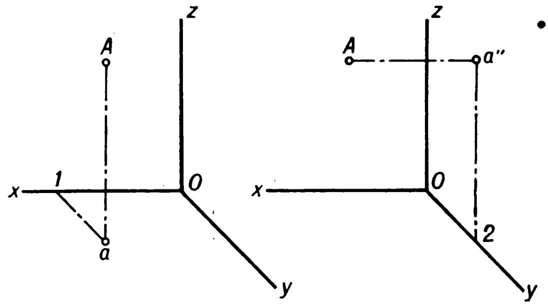


Fig. 24

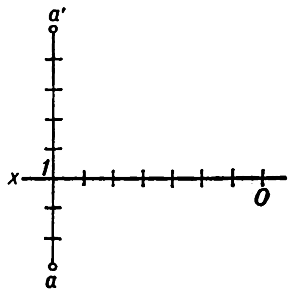


Fig. 25

Figure 16 shows that the construction of the projections of a point are accompanied by the construction of the line segments determining the coordinates of this point if the projection planes are taken for the planes of coordinates. Each of the projections of the point A is defined by two coordinates of this point. For instance, the position of the projection a is defined by the coordinates x and y.

Suppose there given a point A (7, 3, 5); this notation means that the point A is defined by the coordinates $x = 7$, $y = 3$, $z = 5$. If the scale for constructing the drawing is given or chosen, then (see Fig. 25) we take an arbitrary point O on the x-axis and lay off on this axis a line segment $O1$ equal to 7 units, and on the perpendicular through the point 1 the line segments $a1$ equal to 3 units and $a'1$ equal to 5 units. Thus we obtain the projections a and a'. For this construction it is sufficient to take the x-axis only.

Taking the projection axes for the axes of coordinates, we can find the coordinates of the point provided its projections are given. For instance, in Fig. 18 the line segment $O1$ represents the abscissa of the point A, the line segment $a1$ its ordinate, and the line segment $a'1$ its z-coordinate.

If only the abscissa is specified, then we get a plane parallel to the plane defined by the y- and z-axes. Indeed, this plane is a locus of points whose abscissas are equal to the specified quantity (plane P in Fig. 26).

If two coordinates are given, then a straight line is defined which is parallel to the corresponding coordinate axis. For example, having the abscissa and ordinate specified, we obtain a straight line parallel to the z-axis (the straight line AB in Fig. 26). AB is the line of intersection of two planes P and Q, where Q is the locus of points with equal ordinates. The line AB serves as the locus of points with equal abscissas and ordinates.

If all the three coordinates are specified, then a point is defined. Figure 26 illustrates a point K obtained at the intersection of three planes of which P is the locus of points specified by an abscissa, Q is the locus of points specified by an ordinate, and R is the locus of points specified by a z-coordinate.

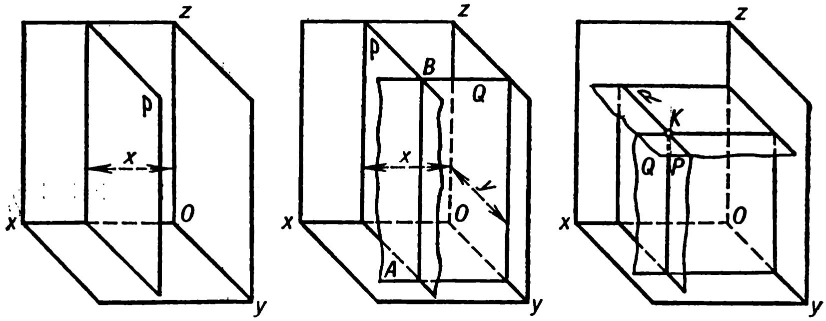


Fig. 26

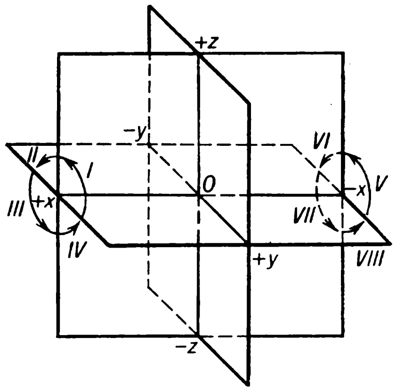


Fig. 27

A point may be located in any of the eight octants which are numbered as in Fig. 27. Therefore we must know not only the distance of a given point from this or that coordinate plane, but also the direction along which this distance should be laid off, for this purpose the coordinates of points are expressed in algebraic numbers. For taking the coordinates we shall use a system of signs indicated in Fig. 27, i.e. we shall use the so-called "right-handed system". The right-handed system is characterized by that the "positive" ray Ox is rotated in the direction of the "positive" ray Oy through an angle of 90° anticlockwise (provided we view the plane xOy from above).

In the system called "left-handed" the "positive" ray Ox is directed to the right from the point O.

When representing solids we usually take for coordinate planes not the planes of projection but a system of some three mutually perpendicular

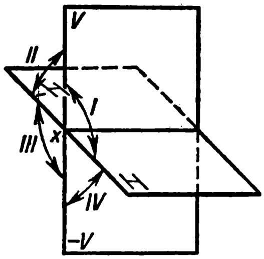


Fig. 28

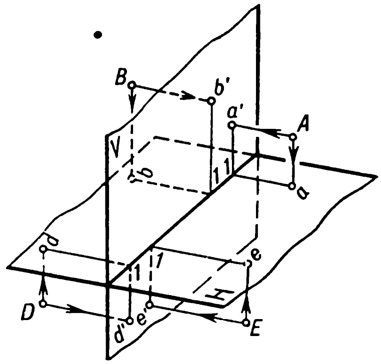


Fig. 29

planes associated directly with a given solid, for instance, the faces of a right parallelepiped, two faces and the plane of symmetry, etc. Such system of coordinates is sometimes termed "inside".

Sec. 7. Points in Quadrants and Octants

In Sec. 4 it was said that, when intersecting, the planes V and H form four dihedral angles which are called *quadrants* or *quarters of space*, which are conventionally numbered as shown in Fig. 28. The axis of projection divides either of the planes H and V into half-planes conventionally designated H and $-H$, V and $-V$. If, for instance, a point is located in the second quadrant, then its horizontal projection is situated on $-H$, and the vertical projection on V.

Henceforward, we shall take the drawing of Fig. 13 as the base for constructing the drawing of a point contained in any of the four quadrants.

When considering orthogonal projections, it is assumed that the viewer is located in the first quadrant at an infinite distance from the planes of projection V and H. The planes of projection are considered to be opaque, therefore visible are only points located in the first quadrant and also on the half-planes V and H.

Figure 13 gives the drawing for the case when a point is situated in the first quadrant (see Fig. 29). If the point is equidistant from V and H, then $a'1 = a1$.

Figure 30 shows a point B located in the second quadrant, i.e. above $-H$ and behind V (Fig. 29). The point B is closer to V than to $-H$: in the drawing $b1 < b'1$. Shown in the same drawing is a point C which is equidistant from $-H$ and V: the projections c' and c coincide.

A point D situated in the third quadrant is shown in Fig. 31. We see its horizontal projection above the axis of projection, while the vertical projection is below the axis. Since $d1 > d'1$, the point D is situated at a greater distance from $-V$ than from $-H$. Figure 32 gives points E and F

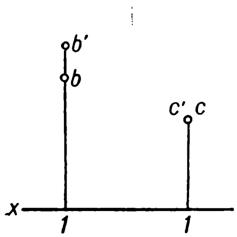


Fig. 30

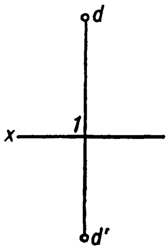


Fig. 31

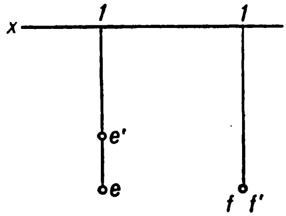


Fig. 32

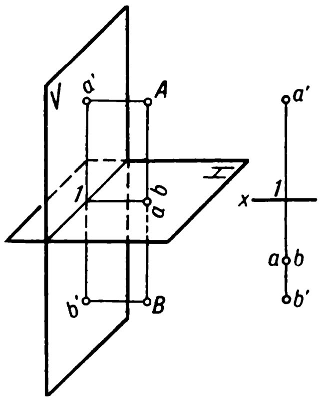


Fig. 33

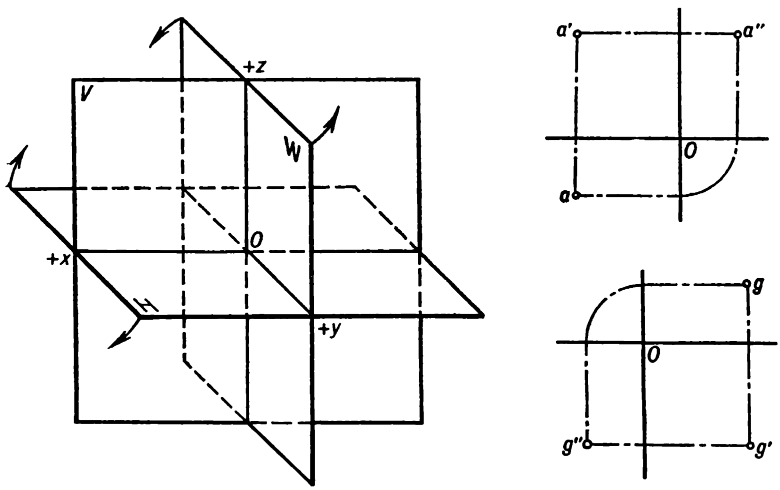


Fig. 34

situated in the fourth quadrant. The point E is nearer to H, than to $-V$; $e'1 < e1$ (Fig. 29). The point F is equidistant from $-V$ and H: $f1 = f'1$.

Figure 33 represents (in the system V, H) points A and B situated symmetrically about the plane H. In the drawing (Fig. 33, right) the horizontal projections of such points coincide, the vertical projections being equidistant from the axis of projection: $a'1 = b'1$.

In the drawing practice use is made of the first and third quadrants. For more detail see Sec. 41.

As is shown in Fig. 27, the coordinate planes, intersecting at right angles, form eight trihedral angles called octants which are numbered according to the drawing. As is seen from Fig. 28, the quadrants are numbered as I to IV octants.

Using for determining coordinates the system of signs indicated in Fig. 27, we get the following table:

Octant	Signs of Coordinates			Octant	Signs of Coordinates		
	x	y	z		x	y	z
I	+	+	+	V	−	+	+
II	+	−	+	VI	−	−	+
III	+	−	−	VII	−	−	−
IV	+	+	−	VIII	−	+	−

For instance, the point $(-20, +15, -18)$ is found in the eighth octant. The planes are brought into coincidence according to Fig. 34, i.e. the plane W is turned anticlockwise if the plane H is viewed in the direction from $+z$ to O.

Figure 34 (right) gives the drawings of two points: A situated in the first octant, and G located in the seventh octant; the projections of one and the same point cannot coincide in these octants. For the rest of the octants two or all the three (for the second and eighth octants) projections of one and the same point can appear to be coincident.

QUESTIONS TO SECS. 6-7

1. What are the rectangular Cartesian coordinates of a point?
2. In what succession are the coordinates written in the notation of a point?
3. What are the quadrants or the quarters of space?
4. What are the octants?
5. What signs have the coordinates of a point located in the seventh octant?

6. What is the difference between the "right-handed" and "left-handed" systems of coordinates?

7. What is the difference between the drawings of two points one of which is located in the first quadrant, and the other in the third?

Sec. 8. Forming Additional Systems of Projection Planes

Until now we have dealt with two systems of projection planes: *V*, *H* and *V*, *H*, *W*. But if necessary, we can form other systems as well. For instance, introducing into the system *V*, *H* a plane *S* perpendicular to the plane *H* (Fig. 35), besides the system *V*, *H* with the projections a' and a of point *A*, we get one more system *S*, *H* with the projections a_s and a of the same point *A*.

Is a new system *V*, *S* also formed in this case? The answer is negative, since the planes *V* and *S* are not perpendicular to each other.

The plane *H* enters both systems *V*, *H* and *S*, *H*. Therefore, the projection a of point *A* refers also to the system *S*, *H* (see Fig. 35). When projecting the point *A* on the plane *S*, we get the point a_s at a distance a_s2 from the plane *H* which is equal to *Aa* and $a'1$.

In Figure 36 the planes *V*, *H* and *S* are shown coincident with the plane of the drawing; the drawing thus obtained is given in Fig. 37. In addition to the axis V/H, one more axis S/H is introduced. This axis is chosen according to the conditions of a concrete assignment. From the point a a line of recall is drawn perpendicular to the axis S/H on which the line segment a_s2 is laid off. This segment is equal to the line segment $a'1$, i.e. to the distance in space from the point *A* to the plane *H*.

Figure 38 shows a drawing in which, along with the system *V*, *H*, one more system (*V*, *T*) is given, i.e. into the system *V*, *H* an additional plane *T* is introduced which is perpendicular to *V*. Now both systems contain the plane *V*. Therefore the distance of the point *A* from this plane is retained, hence in the drawing the line segment a_t2 must be equal to the line segment $a1$.

Obviously, the plane *W* (Fig. 15) may be interpreted as an additional plane drawn perpendicular both to *V* and *H*. In this case, aside from the system *V*, *H* we usually consider the system *V*, *W*. By analogy with Fig. 38, the drawing of Fig. 22 might be transformed into one shown in Fig. 39 (left), where $b''2$ is equal to $b1$. And if we use an auxiliary straight line as in Fig. 17 (the extended bisector of the angle xOz), then the construction takes the form shown in Fig. 39 (right). May we proceed analogously when constructing, for instance, the projection a_s (Fig. 37) or a_t (Fig. 38)? Yes, and it is shown in Fig. 40. But here, of course, the angle equal to 45°, as in Fig. 17, is not obtained. As is seen from the drawings of Fig. 40, we have to draw the bisector of the angle formed by the axes V/H and S/H (Fig. 40, left) and the axes V/H and V/T (Fig. 40, right).

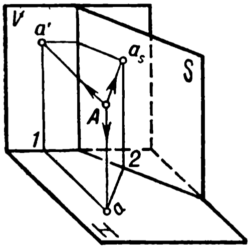


Fig. 35

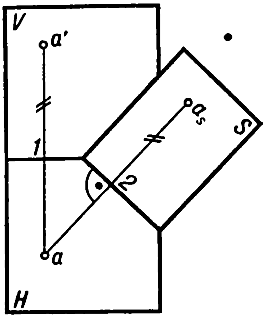


Fig. 36

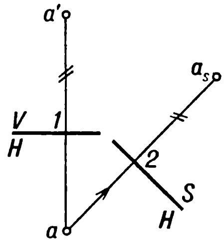


Fig. 37

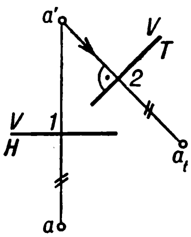


Fig. 38

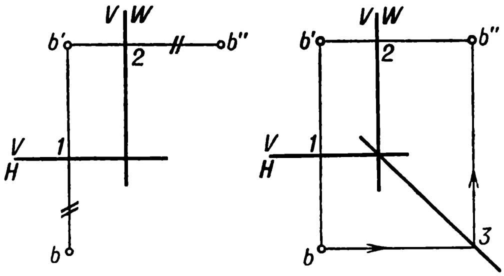


Fig. 39

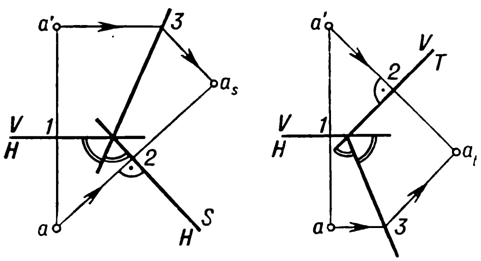


Fig. 40

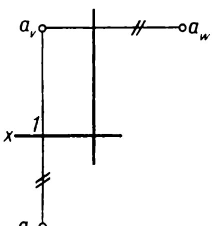


Fig. 41

But, as it was stated in Sec. 5, the constructions shown in Fig. 39 (left) and in Figs. 37 and 38 are preferable.

Below (Sec. 33) we shall come across other examples illustrating the introduction of additional planes for forming a required system of projection planes.

We have denoted by a_s and a_t the projections obtained on the additional planes of projection (say, on S or T). In this connection, it would be appropriate to use the notation a_h, a_v, a_w (as in Fig. 41) instead of a, a', a''. But for the projections on the planes H, V, W we shall use mainly the conventional notation; for instance, a, a', a'' which are the projections of a point A on these planes of projection.

Sec. 9. Drawings Without Axes of Projection

Henceforward, along with the drawings containing the axes of projection, we shall use drawings without indicating the axes.

From the comparison of the drawings given in Fig. 42 it follows that in one case the position of the planes V and H is established by drawing the line of their intersection, and the distances of the point A from these planes are thus defined. Considering the second drawing of Fig. 42, we see that there is no question as to the distances of the point A from the planes V and H, since the axis of projection is absent. Under consideration is a point A specified by its projections irrespective to the location of the projection planes. In this case the line of recall acquires greater importance, therefore it should be carefully constructed.

Is it possible, having a drawing without an indicated axis of projection, to introduce this axis and thus to specify the distances of a point from the planes V and H chosen arbitrarily? Yes, it is possible. When introducing the axis, it should be drawn perpendicular to the line of recall at any point on this line (if there is no specific condition). This leaves the position of the projections unchanged. Indeed, by drawing the axis of projection we choose a certain position of the dihedral angle VH with respect to a given point A (Fig. 43). An upward or a downward displacement of the axis in the drawing corresponds to a translation in space of the dihedral angle VH in the direction of the bisector plane of the dihedral angle* adjacent to the angle VH (V_1H_1 in Fig. 43 is the new position of the dihedral angle VH).

The introduction of the projection axis (which is usually done in accordance with a certain condition) was shown in Figs. 37 and 38: the axes S/H and V/T. They were needed for construction purposes: from them the dimensions were measured. If considered in their initial meaning, i.e. as the lines of intersection of the projection planes, the axes in general help to form a three-dimensional picture from the drawing.

Reference bases for measuring the dimensions are essential components of mechanical drawings. The choice of the bases is not restricted, their positions being determined proceeding from necessity and advisability.

Figure 44,*a* shows how to find the difference between the distances of the points A and B from the projection planes H, V, and W. In Fig. 44,*b* the left drawing is given with the projection axes, the right one without them.

In this example the difference between the distances of the points from

*The *bisector plane of a dihedral angle* is a plane passing through the edge of the dihedral angle and bisecting it.

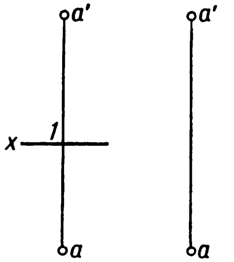


Fig. 42

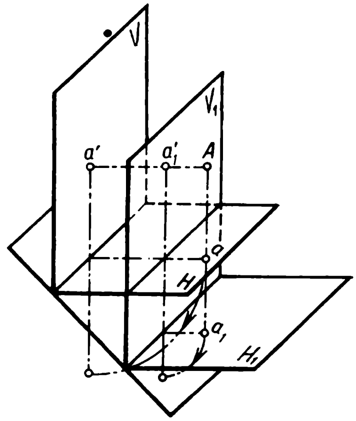
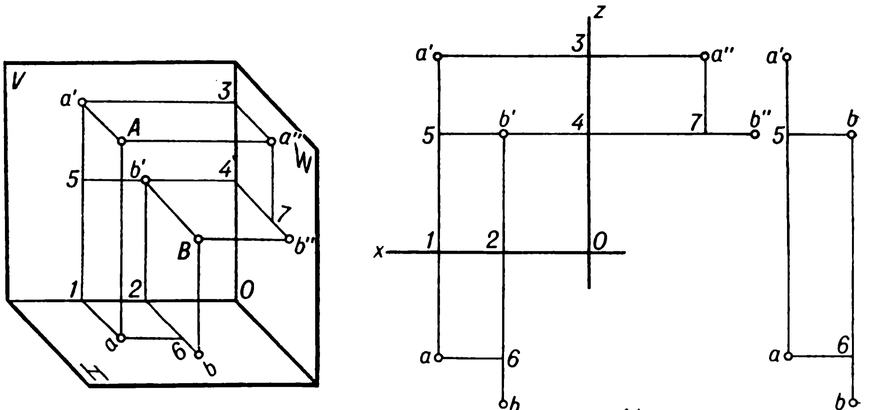


Fig. 43



(a) (b)

Fig. 44

the plane H is determined by the line segment $a'5$ equal to $a'1-b'2$ or to $a''7$; from the plane V by the line segment $b6$ equal to $b2-a1$ or to $b''7$; from the plane W by the line segment $b'5$ equal to $a'3-b'4$ or $a6$.

QUESTIONS TO SECS. 8-9

1. How is a system of projection planes formed?

2. What condition must be satisfied by the plane introduced into the system V, H as an additional plane of projection?

3. How is the projection of a point specified in the system V, H constructed in the plane S which is perpendicular to the plane H?

4. Can we determine the distances of a point from the projection planes in the presence of the axis of projection?

5. How should a drawing be understood in the absence of the axis of projection?

6. What is the purpose of the axes S/H and V/T in Fig. 40?

7. How is the distance of a point from the planes H and V determined in the drawing in the V, H system?

Sec. 10. The Projections of a Line Segment

Suppose we are given the vertical and horizontal projections of points A and B (Fig. 45). Joining the like projections of these points, we get the vertical ($a'b'$) and the horizontal (ab) projections of the line segment AB (see Sec. 2, Item 5).

May we assert that this drawing (Fig. 45) represents just a line segment? The answer is "Yes". If we imagine that through $a'b'$ and ab projecting planes (i.e. planes perpendicular to V and H, respectively) are drawn, then they will intersect along a straight line, AB being its segment (Fig. 46). Furthermore, a point specified by its projections on $a'b'$ and ab belongs to the line segment AB.

Given in Fig. 47 is a drawing of a segment AB in the system V, H, W. The projections a'' and b'' are constructed in the way illustrated in Fig. 18 for one point A.

The points A and B are located at different distances from each of the planes V, H, W, i.e. the line AB is parallel to none of them, none of the projections of the line being parallel or perpendicular to the axis of projection. Such a straight line is termed an *oblique line.*

Each of the projections is shorter than the segment itself: $a'b' < AB$, $ab < AB$, $a''b'' < AB$. Denoting the angles between the line and the planes H, V, and W by α, β and γ, respectively, we get

$$ab = AB \cos \alpha, \quad a'b' = AB \cos \beta, \quad a''b'' = AB \cos \gamma,$$

If $ab = a'b' = a''b''$, then the line forms equal angles ($\approx 35°$) with the projection planes (see Sec. 13), each of the projections of the line being inclined to the corresponding axes of projection or to the lines of recall at an angle of 45°.

Indeed, if (Fig. 48) $a'b' = ab$ and $a'b' = a''b''$, then the figure $a'b'ba$ is an equilateral trapezium and $b'1 = b2$, whence $b''3 = a''3$, i.e. the angle $a''b''3 = 45°$, and since the figure $a'b'b''a''$ is a parallelogram, either of the angles $b'a'1$ and $ba2$ is equal to 45°.

How is, for instance, the profile projection of a line segment constructed on a drawing having no axes of projection? The construction is shown in Fig. 49, where, along with the initial drawing of the segment AB of an obli-

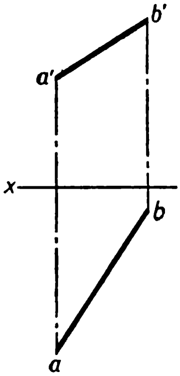


Fig. 45

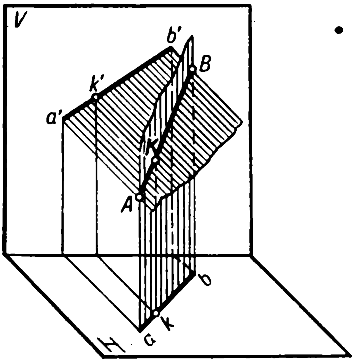


Fig. 46

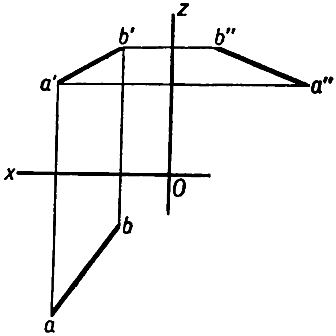


Fig. 47

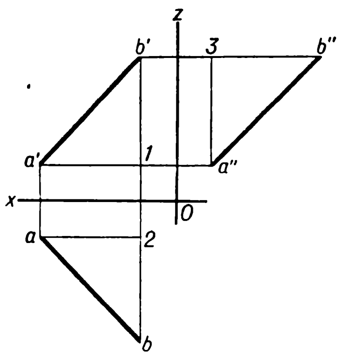


Fig 48

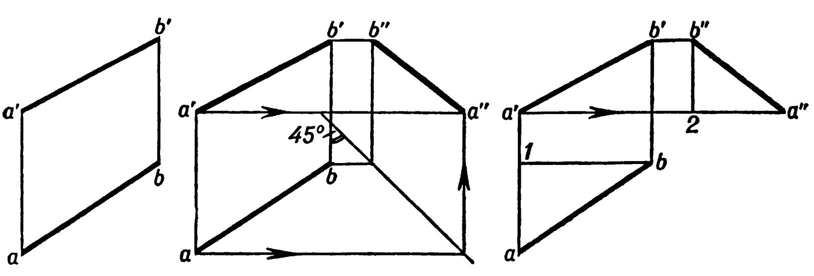


Fig. 49

que line (left), we see the use of an auxiliary line drawn at an angle of 45° to the direction of the line of recall *b'b* (middle), and the construction using the difference between the distances of the points *A* and *B* from the plane *V*, i.e. the line segment *a1* (right). Given, say, the projection *a''* (on the line of recall *a'a''*), we lay off *a''2* equal to *a1* and find the position of the projection *b''* by drawing from the point *2* a perpendicular to the line of recall joining *b'* and *b''*.

Sec. 11. Particular Positions of a Straight Line Relative to the Projection Planes

A straight line may occupy particular positions relative to the projection planes. Let us consider them according to the following two characteristics:

A. A line is parallel to one projection plane.

B. A line is parallel to two projection planes.

In the first case one of the projections of the line segment is equal to the segment itself. In the second case two projections of the line segment are equal to the segment itself*.

A. A Line Parallel to One Projection Plane

1. A line is parallel to the plane *H* (Fig. 50). In this case the vertical projection of the line is parallel to the axis of projection and the horizontal projection of a segment of this line is equal to the segment itself: $ab = AB$. Such line is called *horizontal.*

If, for instance, the projection *a'b'* coincides with the axis of projection, then the line segment *AB* is contained in the plane *H*.

2. A line is parallel to the plane *V* (Fig. 51). In this case its horizontal projection is parallel to the axis of projection and the vertical projection of the segment of this line is equal to the segment itself: $c'd' = CD$. This line is called *vertical.*

If, for instance, the projection *cd* coincides with the axis of projection, then this corresponds to the position of the segment *CD* contained in the plane *V*.

3. A line is parallel to the plane *W* (Fig. 52). In this case the horizontal and vertical projections of the line are situated on a single perpendicular to the axis of projection *Ox*, and the profile projection of this line is equal to the segment itself: $e''f'' = EF$. This line is termed *profile.*

May we consider that the drawings like those shown in Figs. 50 and 51 represent segments of straight lines? Yes; the proof is the same as for an oblique line (Fig. 46).

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*Of course, taking into consideration the scale of a drawing.

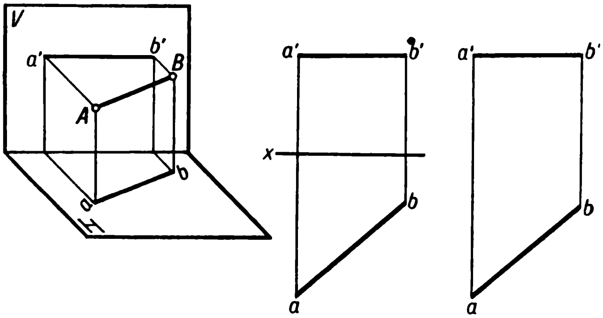

Fig. 50

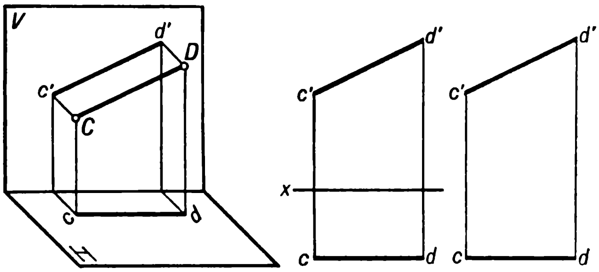

Fig. 51

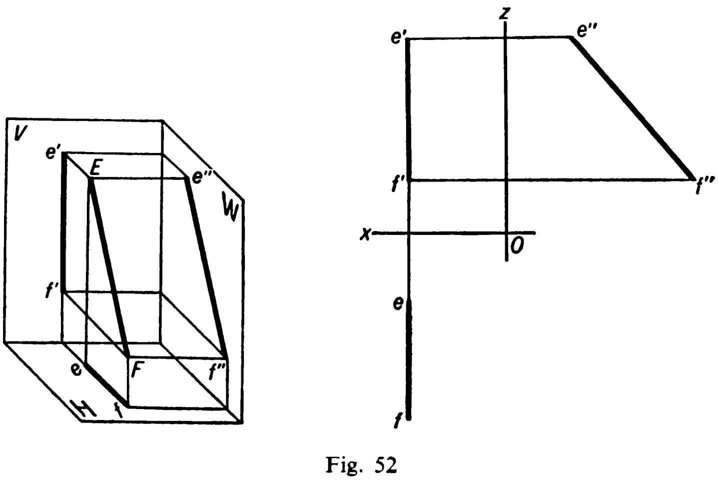

Fig. 52

But if in a drawing made in the system *V*, *H* both projections are perpendicular to the axis of projection, then the projecting planes drawn through *ef* and *e'f'* merge into one plane. In this case the original may be not only a straight line, but a plane curve as well (see Fig. 53).

B. A Line Parallel to Two Projection Planes

1. A line is parallel to the planes *V* and *H* (Fig. 54), i.e. perpendicular to the plane *W*. The projection of the line on the plane *W* represents a point.

2. A line is parallel to the planes *H* and *W* (Fig. 55), i.e. perpendicular to the plane *V*. The projection of the line on the plane *W* represents a line-segment equal to *cd*.

3. A line is parallel to the planes *V* and *W* (Fig. 56), i.e. perpendicular to the plane *H*. The projection of the line on the plane *W* will be a line segment parallel and equal to *e'f'*.

Figure 57 shows the positions of the considered lines*.

The projections of line segments are usually constructed with the end-points indicated. If for this or that reason an indefinite portion of a straight line is shown, then, practically, also a line segment is represented, but in this case its end-points are not indicated. In this event we may designate each projection only by one letter referring it to a point on the line (Fig. 58): "a straight line passing through the point *A*".

Let us consider the left drawing of Fig. 59. About the line shown in it we may say only that it passes through the point *L* and is parallel to the plane *H*. But in all other respects the position of this line is not defined. It would be completely defined if the horizontal projection were given, i.e. a projection on the line with respect to which the given line is parallel.

But if we have a straight line specified by two its points (for instance, a line segment specified by its end-points), then it is possible to determine the exact position of this line even if its projection on the plane to which it is parallel is not given. For example, if a line segment *AB* is given (Fig. 59, right), then we can determine not only parallelism of this line to the plane *H*, but also the fact that the point *A* is farther from the plane *V* as compared with the point *B*.

Sec. 12. A Point on a Straight Line. Traces of a Line

Figure 60 shows the drawing of an oblique line passing through the point *A*. If it is known that the point *B* belongs to this line and that the horizontal projection of the point *B* is situated at point *b*, then the projection *b'* is determined in the way shown in Fig. 60.

In Figure 61 a point is constructed on a profile line. Suppose the projection *c'* of this point is given and it is required to find its horizontal projec-

*These straight lines are sometimes called "projecting lines".

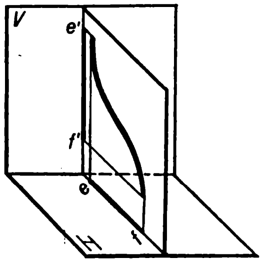

Fig. 53

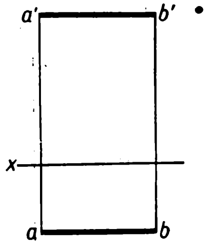

Fig. 54

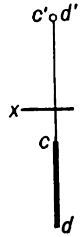

Fig. 55

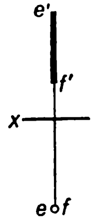

Fig. 56

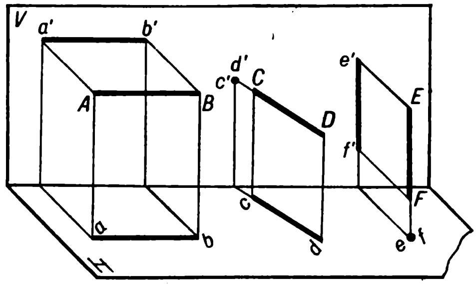

Fig. 57

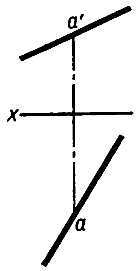

Fig. 58

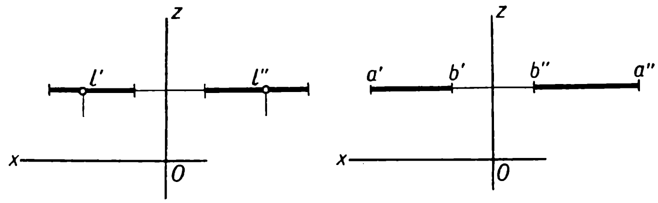

Fig. 59

tion. The construction is carried out with the aid of the profile projection $a''b''$ of the line segment AB taken on a profile line. All necessary constructions are shown in the figure by arrows. We first determine the projection c'' and then the required projection c.

One of the properties of parallel projection is that the *ratio of line segments is equal to the ratio of their projections* (Fig. 62): $\frac{AC}{CB} = \frac{a_p c_p}{c_p b_p}$, since Aa_p, Cc_p, and Bb_p are parallel lines. Analogously, the ratio of the line segments on the projection of a line is equal to the ratio of the line segments on this line. If a point bisects a line segment, then the projection of this point also bisects the projection of the line segment, and vice versa.

Whence it follows that the division of the projections $a'b'$ and ab by the points c' and c, respectively, shown in Fig. 61 corresponds to the division

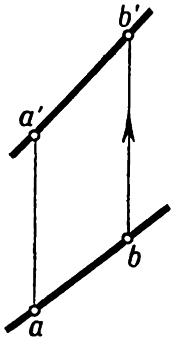

Fig. 60

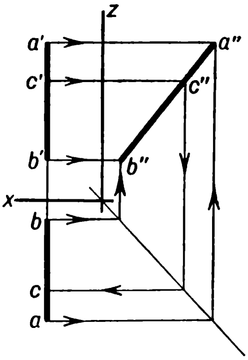

Fig. 61

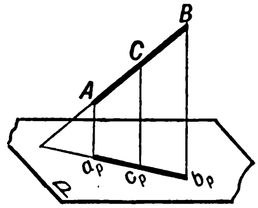

Fig. 62

of the line segment *AB* in space by the point *C* in the same ratio. This may be used for a simpler construction of a point on a profile line. If (the same as in Fig. 61) the projection *c'* is given on the projection *a'b'* (Fig. 63), then, obviously, we have to divide *ab* in the same ratio in which the point *c'* divides the projection *a'b'*. Drawing from the point *a* an auxiliary line, we lay off on it *a1* = *a'c'* and *1-2* = *c'b'*. We draw then a straight line *b2* and through the point *1* a straight line to intersect *ab* at point *c*. This point represents the required horizontal projection of the point *C* belonging to the line segment *AB*.

Figure 64 illustrates an example of dividing a line segment in a given ratio.

The line segment *CD* is divided in the ratio 2 : 5. An auxiliary line is drawn from the point *c* on which seven (2+5) equal segments of an arbitrary length are laid off. Drawing the line segment *d7* and through the point *2* a straight line parallel to it, we get a point *k*, and *ck* : *kd* = 2 : 5; then we find *k'*. The point *K* divides the line segment *CD* in the ratio 2 : 5.

Figure 65 shows points *M* and *N* at which a straight line specified by the segment *AB* intersects the planes of projection. These points are called the *traces:* point *M* is the *horizontal trace of the line*, and point *N* its *vertical trace.*

The horizontal projection of the horizontal trace (point *m*) coincides with the trace itself, and the vertical projection of this trace *m'* lies on the axis of projection. The vertical projection of the vertical trace *n'* coincides with the point *N*, and the horizontal projection *n* lies on the same axis of projection.

Consequently, to find the horizontal trace we have (see Fig. 66) to extend the vertical projection *a'b'* to intersect the *V*/*H* axis and to draw a perpendicular through the point *m'* (which is the vertical projection of the horizontal trace) to the *V*/*H* axis to intersect the extended horizontal projection *ab*. Point *m* is the horizontal projection of the horizontal trace; it coincides with the trace itself (≡ is the sign of coincidence).

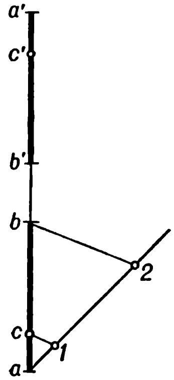

Fig. 63

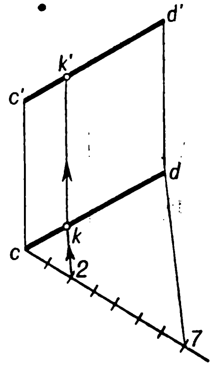

Fig. 64

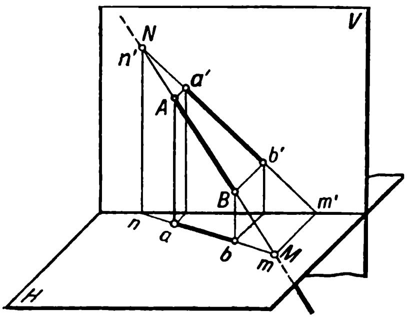

Fig. 65

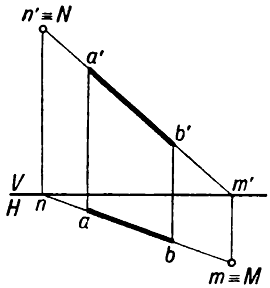

Fig. 66

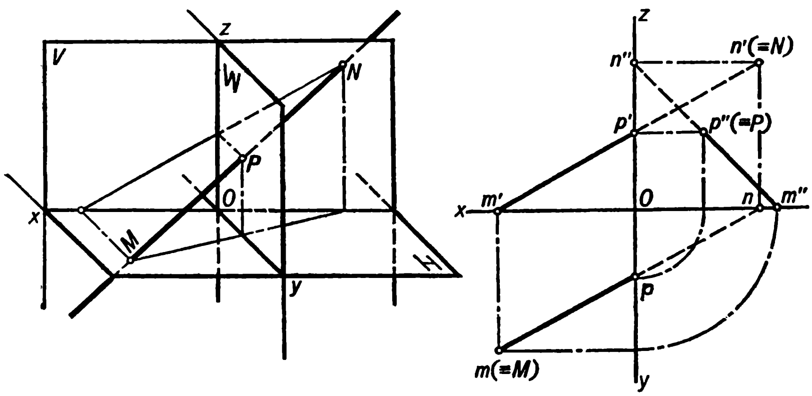

Fig. 67

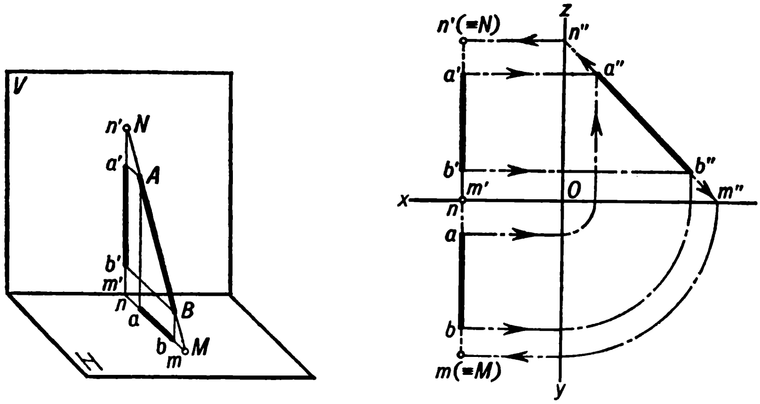

Fig. 68

The vertical trace is found in the following way: extend the horizontal projection *ab* to intersect the *V*/*H* axis; through the point *n* (the horizontal projection of the vertical trace) draw a perpendicular to intersect the extended vertical projection *a'b'*. The point *n'* thus obtained is the vertical projection of the vertical trace; it coincides with the trace itself.

By the positions of the points *M* and *N* we can judge to which quadrants the given line refers. In Figure 65 the straight line *AB* passes through the fourth, first, and second quadrants.

A straight line has no trace in a plane of projection if it is parallel to this plane.

In Figure 67 a straight line pierces not only the planes *H* and *V*, but also the plane *W*. Point *P* is the *profile trace* of the line, i.e. the trace on the profile plane of projection. This trace coincides with its own projection on the plane *W*, its vertical and horizontal projections lying on the *z*- and *y*-axes, respectively.

In this case the line passes beyond the point *P* through the fifth octant, and meeting then the plane *V*, enters the sixth octant; leaving the first octant, the line enters the fourth octant*.

The corresponding drawing is given in Fig. 67 (right). The straight line is shown in the first octant (the projections *mp*, *m'p'*, and *m''p''*) and in the fifth octant (the projections *pn*, *p'n'* and *p''n''*).

If the projection planes are taken for the coordinate planes, then the coordinate *z* of the horizontal trace of the line is equal to zero, the coordinate *y* of the vertical trace is equal to zero, and the coordinate *x* of the profile trace is equal to zero.

---

*Let us agree here to use continuous lines for those projections of drawings which correspond to the position of a line segment located in the first quadrant or in the first octant.

The traces of a profile line (Fig. 68) can be constructed in the following way (Fig. 68, right): we construct the profile projection ($a''b''$), determine the positions of the profile projections of the horizontal trace (m'') and the vertical trace (n''), and then find the positions of the remaining projections of these traces (the successive stages of construction are shown by arrows).

QUESTIONS TO SECS. 10-12

1. What is the position of an oblique line relative to the projection planes?
2. How is it proved that a drawing containing two interconnected projections in the shape of line segments represents exactly a line segment?
3. How is the relation between a projection of a line segment and the line segment itself expressed?
4. How is a straight line situated in the system V, H, W if all three projections of its segment are equal in length?
5. How to construct the profile projection of a segment of an oblique line given its vertical and horizontal projections?
6. How is the construction of the preceding question carried out on a drawing without the axes of projection?
7. What positions of a straight line in the system V, H, W are considered to be particular?
8. What is the position of the vertical projection of a line segment if its horizontal projection is equal to the line segment itself?
9. What is the position of the horizontal projection of a line segment if its vertical projection is equal to the line segment itself?
10. What is the property of parallel projection concerning the ratio of line segments?
11. How is a line segment divided on the drawing in a given ratio?
12. What is the trace of a straight line on a projection plane?
13. What coordinate is equal to zero: (a) for a vertical trace of a straight line, (b) for a horizontal trace of a straight line?
14. Where is the horizontal projection of a vertical trace situated?
15. Where is the vertical projection of the horizontal trace of a straight line located?
16. Is it possible for a straight line in the system V, H, W to have traces on each of these planes merging into one point?

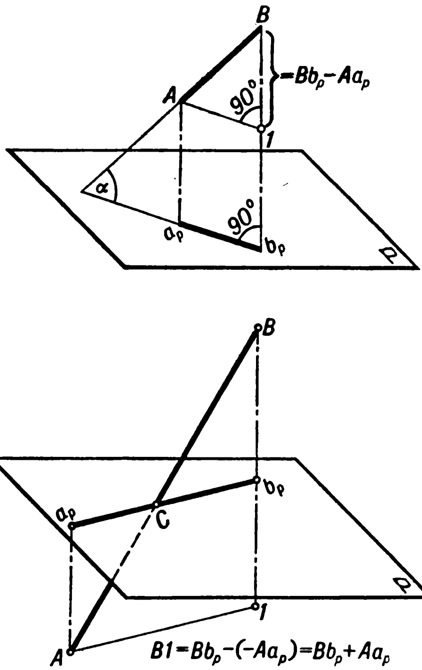

Fig. 69

Sec. 13. Constructing the True Length of a Segment of an Oblique Line and the Angles of Inclination of a Straight Line to the Projection Planes *V* and *H*

Examining the top picture of Fig. 69 we may conclude that the line segment AB is the hypotenuse of a right triangle $AB1$ in which one leg is equal to the projection of the line segment ($A1 = a_pb_p$), and the other leg is equal to the difference of the distances of the end-points of the segment from the projection plane P.

If the coordinates defining the distances of the end-points of the segment from the projection plane have different signs (Fig. 69, bottom), then an algebraic difference is meant:

$$B1 = Bb_p - (-Aa_p) = Bb_p + Aa_p.$$

The angle between a straight line and a projection plane is defined as an angle formed by the line with its projection on this plane. This angle is one of the interior angles of the right triangle constructed for determining the true length of the line segment.

Obviously, knowing from the drawing its legs, we can construct the triangle at any place of the drawing area. Figure 70 illustrates the construction applied by G. Monge: laid off from point *1* is a segment $a_1'1$ equal to the projection ab, and the hypotenuse $a_1'b'$ representing the true length of the segment AB is drawn. The angle with the vertex at point a_1' is equal to the angle between AB and the plane H.

In Figure 71 (left) the length of the line segment AB and the angle formed by the line AB with the plane H are determined from a right triangle constructed on the projection ab with $b\bar{B}$ as its second leg which is equal to $b'1$. As is obvious, $AB = a\bar{B}$.

In Figure 71 (right) the length of the line segment and the angle formed by the line AB with the plane V are determined from the right triangle constructed on the projection $a'b'$ ($a'\bar{A} = a2$). $AB = b'\bar{A}$.

Must the angles α and β satisfy any condition in case of an oblique line? Yes, either of them must be acute. Moreover, for an oblique line $\alpha+\beta < 90°$. Indeed, (Fig. 72) as is obvious from the right triangle $n'm'm$, $\delta+\beta = 90°$. But in the triangles $n'm'm$ and $n'nm$ having a common hypotenuse $n'm$ the leg $n'm'$ is longer than the leg $n'n$ and, consequently, $\delta > \alpha$. Substituting α instead of δ into $\delta+\beta = 90°$, we get $\alpha+\beta < 90°$.

Consider right-angled triangles $ab\bar{B}$ and $b'a'\bar{A}$ (Fig. 71). In either of them the hypotenuse represents the true length of a line segment, one of the legs being the projection of this segment. The other leg is equal to the difference of the distances of the end-points of the segment from the corresponding plane of projection ($b\bar{B} = b'1$ = the difference of the distances from H, and $a'\bar{A} = a2$ = the difference of the distances from V). Besides, one of these triangles contains the angle between the segment and the plane H (the angle α), and the other the angle between the segment and the plane V (the angle β).

In this case we have determined the hypotenuse and the angle, knowing the legs of a triangle. But we may come across such a situation: given the hypotenuse and the angle, required: to determine the legs (i.e. *given the true length of a segment and the angles formed by it with the projection planes; it is required to construct the projections of this segment*).

Suppose (Fig. 73) that AB is a given line segment (it corresponds to the hypotenuses $a\bar{B}$ and $b'\bar{A}$ in Fig. 71). We construct on it as on the diameter a circle. Taking the point A for the vertex, we construct the angle α (i.e. the given angle with the plane H) and the right triangle $A1B$. From the comparison of this triangle with the triangle $ab\bar{B}$ (Fig. 71) it follows that the leg $A1$ represents the horizontal projection of the segment AB, while the leg $B1$ the difference between the distances of the end-points of the segment AB from the plane H.

Let us also construct (Fig. 73) the right triangle $A2B$ using the same hypotenuse AB and the given angle β with the projection plane V, and compare it with the triangle $b'a'\bar{A}$ shown in Fig. 71. Obviously, the leg $B2$ represents the vertical projection of the given segment, and the leg $A2$ the difference between the distances of the end-points of the segment from the plane V.

Let us now make a drawing (Fig. 74). Laying off on the line of recall

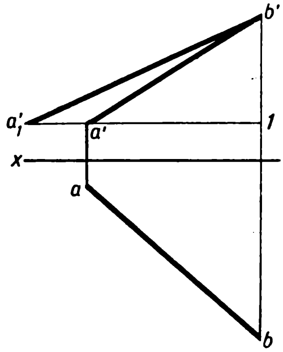

Fig. 70

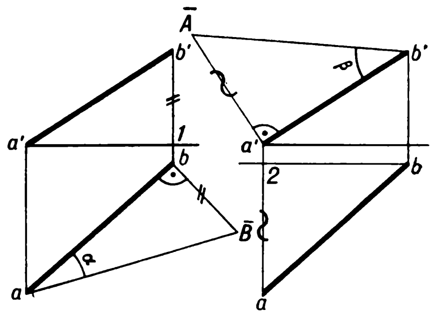

Fig. 71

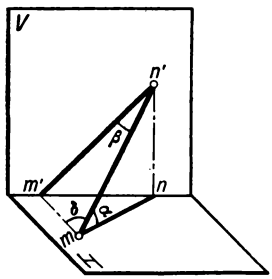

Fig. 72

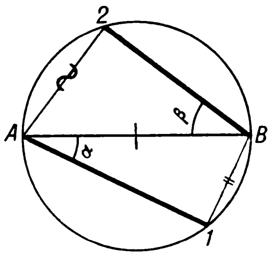

Fig. 73

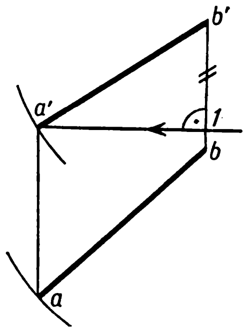

Fig. 74

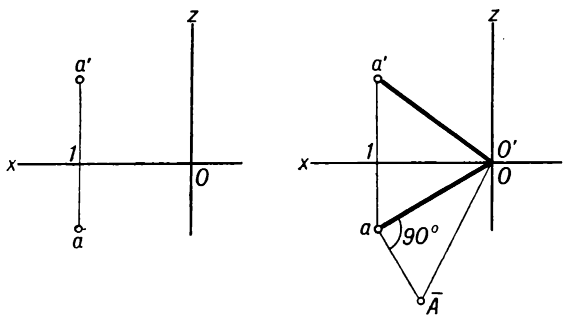

Fig. 75

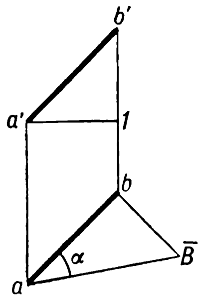

Fig. 76

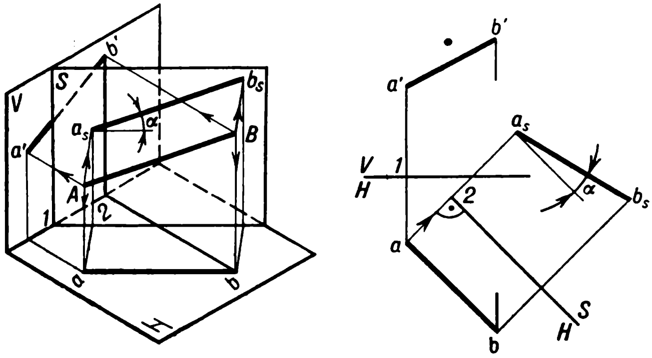

Fig. 77

$b'b$ from the point b' a line segment $b'1$ equal to $B1$ (see Fig. 73), we draw through point 1 a straight line perpendicular to $b'b$. Intersecting this line with an arc described from b' as centre (its radius must be equal to the vertical projection, i.e. to the segment $B2$), we get the point a'. To find the horizontal projection a we intersect the line of recall drawn through the point a' by an arc whose radius is equal to $A1$ (see Fig. 73). In this case the following must be obtained: $a'a-b1 = A2$.

Figure 74 gives only one position of the line segment. There are seven other positions at the initial point B. The reader is welcome to represent the segment AB in all these positions as well.

An example of determining the distance from the point A to the point O is given in Fig. 75. First the projections $a'o'$ and ao of the required segment are constructed (the point O is represented by its projections o' and o). Then the triangle $oa\bar{A}$ is constructed one of whose legs is the projection oa, and the other the line segment $a\bar{A}$ equal to $a'1$. The required distance is determined by the hypotenuse $o\bar{A}$.

Now we can determine the angle formed by the straight line inclined at equal angles to the planes H, V, and W with these planes. This angle was considered in Sec. 10 and its magnitude ($\approx 35°$) was indicated. It can be determined, for instance, from Fig. 76: the projections $a'b'$ and ab are equal to each other, and either of the angles $a'b'1$ and $a'ab$ is equal to 45° (see Sec. 10).

The required angle is determined from the right-angled triangle $ab\bar{B}$ in which the leg $b\bar{B} = b'1$. If we put $b'1$ to be equal to unity then $ab = a'b' = \sqrt{2}$ and the angle $\alpha \approx 35°15'$. The angles between this line and the planes V and W are equal to the angle α.

If we supplement the system V, H with the system S, H (see Sec. 8), taking S perpendicular to H and parallel to the line segment given on the drawing, then, obviously, the projection of this segment on the plane S will represent its true length and the angle with the plane H.

Suppose (Fig. 77) it is required to determine the true length of the line

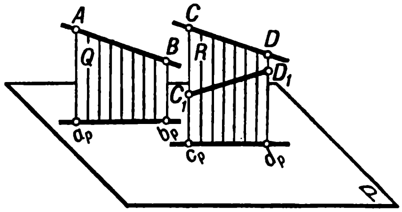

Fig. 78

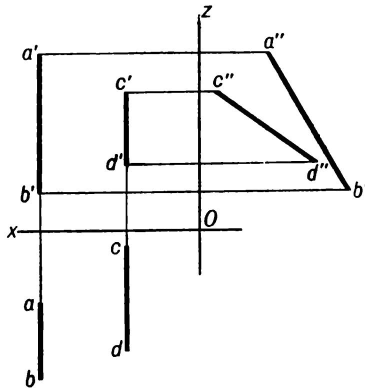

Fig. 79

segment AB and its angle with the plane H. The system V, H is supplemented with a plane S perpendicular to H so that S is parallel to AB. Thus, we have an additional system S, H in which AB is parallel to S (the axis S/H is parallel to ab); in this case the projection a_sb_s represents the true length of the line segment AB.

Sec. 14. The Relative Positions of Two Straight Lines

Parallel Lines. One of the properties of parallel projecting reads: *the projections of two parallel straight lines are parallel to each other.* If (Fig. 78) the straight line AB is parallel to the line CD, then the projecting planes Q and R are parallel to each other. The intersection of these planes with the projection plane P yields the projections a_pb_p and c_pd_p which are parallel to each other.

But though a_pb_p is parallel to c_pd_p (Fig. 78) the straight lines for which a_pb_p and c_pd_p are the projections are not necessarily parallel to each other: for instance, the line AB is not parallel to the line C_1D_1.

From the mentioned property of parallel projecting it follows that the *horizontal projections of parallel lines are parallel to each other, their vertical projections are parallel to each other, and their profile projections are parallel to each other.*

Is the converse true, i.e. will two lines in space be parallel if their like projections on the drawing are pairwise parallel? Yes, if we are given parallel projections of these lines on each of the three projection planes: H, V, and W. But if we are given parallel projections of the lines only on two projection planes, then by this the parallelism of straight lines in space is always verified for oblique lines, and may be verified for lines which are parallel to one of the projection planes.

An example is given in Fig. 79. Though the profile lines AB and CD are specified by the projections ab, $a'b'$ and cd, $c'd'$ which are pairwise parallel,

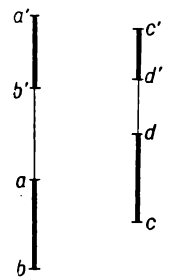

Fig. 80

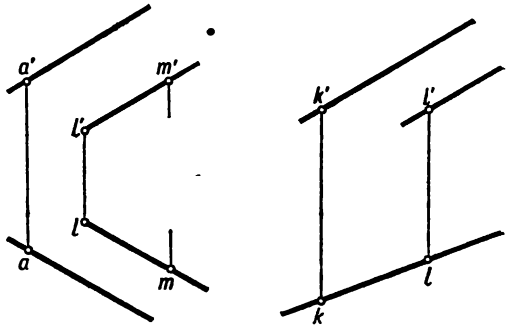

Fig. 81

but the lines themselves are not parallel, which is seen from the relative positions of their profile projections constructed by the given projections.

Thus, *the problem was solved with the aid of the projections of the given lines on the projection plane relative to which the lines are parallel.*

Figure 80 illustrates the case when it is possible to find out that the profile lines *AB* and *CD* are not parallel to each other without resorting to construction of the third projection: it is sufficient to draw one's attention to the interchange of the designating letters.

If it is required to draw through a given point *A* a straight line parallel to a given line *LM*, then (see Fig. 81, left) the construction is reduced to drawing through the point a' a line parallel to $l'm'$, and through the point a a line parallel to lm.

In the case shown in Fig. 81 (right) two parallel lines are contained in a common projecting plane perpendicular to the plane *H*. That is why the horizontal projections of these lines are situated on the straight line.

Intersecting Lines. *If straight lines intersect, then their like projections intersect at a point which is the projection of the point of intersection of these lines.*

Indeed, if the point *K* (Fig. 82) belongs to both lines *AB* and *CD*, then the projection of this point must be the point of intersection of the projections of these lines.

The conclusion that the lines given in the drawing intersect may be drawn always with respect to *oblique lines* irrespective of the fact whether the projections are given on three or two projection planes. The necessary and sufficient condition in this case is only that *the points of intersection of the like projections must lie on a single perpendicular to the corresponding axis of projection* (Fig. 83), *or in case of a drawing without the projection axis* (Fig. 84), *these points must lie on the appropriate line of recall.* But if one of the given lines is parallel to a projection plane and the drawing has no projections of the lines on this plane, then we have no right to assert that such lines intersect even if the above-stated condition is fulfilled. For instance, in Fig. 85 the lines *AB* and *CD*, of which *CD* is parallel to the plane *W*,

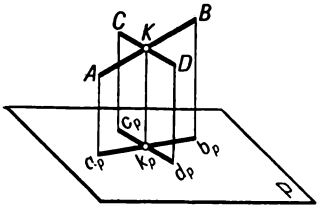

Fig. 82

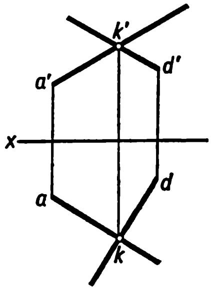

Fig. 83

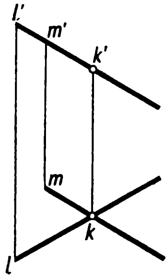

Fig. 84

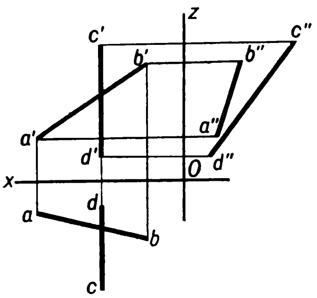

Fig. 85

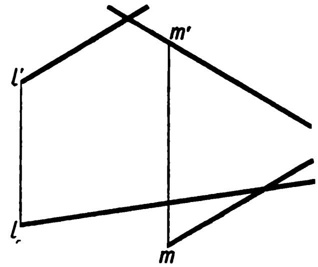

Fig. 86

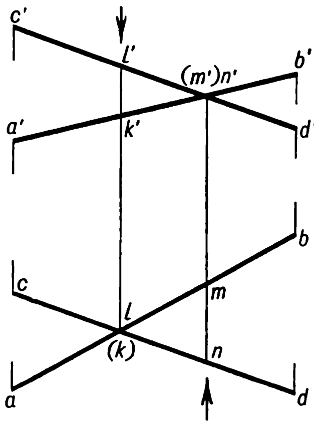

Fig. 87

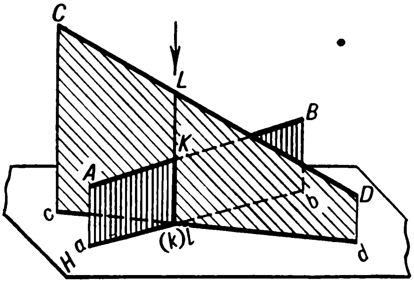

Fig. 88

do not intersect. This can be proved by constructing the profile projections or by applying the rule for dividing line segments in a given ratio.

The intersecting lines shown in Fig. 84 are contained in a common projecting plane perpendicular to the plane V. That is why the vertical projections of these lines are located on one straight line.

Skew Lines. Skew lines do not intersect and are not parallel to each other. Figure 86 shows two skew lines neither of which is parallel to any plane of projection. Although their like projections intersect, but the points of their intersection cannot be joined with a line of recall parallel to the lines of recall $l'l$ and $m'm$, i.e. the lines do not intersect. The lines shown in Figs. 79, 80, and 85 are also skew lines.

How must we consider the point of intersection of the like projections of skew lines? It represents the projections of two points one of which belongs to the first line, and the other to the second of these skew lines. For instance, in Fig. 87 the point with the projections k' and k belongs to the line AB, and the point with the projections l' and l to the line CD. These points are equidistant from the plane V, but their distances from the plane H are different: the point represented by the projections l' and l is farther from the plane H than the point represented by the projections k' and k (see Fig. 88).

The points with the projections m', m and n', n are equidistant from the plane H, but their distances from the plane V are different.

The point represented by the projections l' and l and belonging to the line CD hides the point on the line AB with the projections k' and k when projected on the plane H. The corresponding direction of viewer's sight is indicated with an arrow. When projected on the plane V, the point represented by the projections n' and n and belonging to the line CD hides the point with the projections m' and m on the line AB; the direction of the viewer sight is indicated with an arrow below.

The projections of "hidden" points are designated by the corresponding letters in parentheses.

The points belonging to skew lines and situated on a single projecting line are sometimes called "competing".

Sec. 15. The Projections of Plane Angles

1. *If a plane containing an angle is perpendicular to a projection plane, then the angle is projected on this plane in the form of a straight line.*

2. *If a plane containing a right angle is not perpendicular to a projection plane and at least one of its sides is parallel to this plane, then the right angle is projected on it in the form of a right angle.*

Suppose the side CB of the right angle ACB is parallel to a projection plane (Fig. 89). In this event the line CB is parallel to c_pb_p. Let the second side (AC) of the right angle intersect its projection a_pc_p at point K. We draw in the projection plane through the point K a straight line parallel to c_pb_p. The line KL is also parallel to CB, and the angle CKL turns out to be a right one. According to the theorem on three perpendiculars, the angle c_pKL is also a right one*. Consequently, $a_pc_pb_p$ is also a right angle.

Two converses correspond to this theorem on projecting a right angle (items 3 and 4).

3. *If the projection of a plane angle represents a right angle, then the projected angle will be a right one, provided at least one side of this angle is parallel to a projection plane.*

4. *If the projection of an angle one of whose sides is parallel to a projection plane represents a right angle, then the projected angle is also a right one.*

With the above-stated theorems in mind, we can determine that the angles represented in Fig. 90 are right angles in space.

In what case do the projections of a right angle on two projection planes represent right angles? It happens when one side of the right angle is perpendicular to the third plane of projection (then its other side is parallel to this plane). An example is given in Fig. 91: the side AC is perpendicular to W, the side BC being parallel to W.

Using the knowledge of projecting a right angle, of supplementing the system V, H with a system S, H (Sec. 8), and of the positions of the projections of a line parallel to one of the projection planes (Sec. 11), we can accomplish the following construction: through a point A draw a straight line so that it intersects the given line at right angles. The solution is shown in Fig. 92: the initial position (left), forming one more system (S, H) in addition to V, H, the plane S being parallel to BC (middle), the construction of the line AK perpendicular to BC (right).

Since the plane S is parallel to BC which is provided by drawing the axis S/H parallel to bc, the right angle AKB (or AKC) is projected on the plane S true shape, i.e. in the form of the right angle $a_sk_sb_s$. On constructing the projections of the point A and the line BC on the plane S, we draw a_sk_s perpendicular to b_sc_s, and then obtain the projections k and k', and the projections ak and $a'k'$ (the course of construction is indicated by arrows).

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* According to the three perpendiculars theorem: if $KL \perp c_pK$, then $KL \perp CK$; according to the converse: if $KL \perp CK$, then $KL \perp c_pK$.

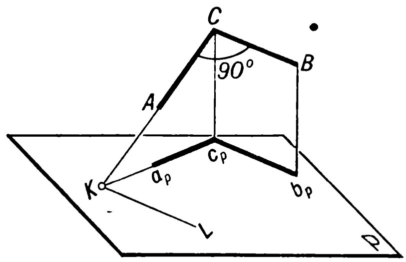

Fig. 89

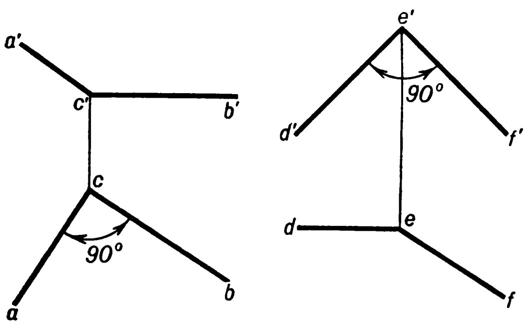

Fig. 90

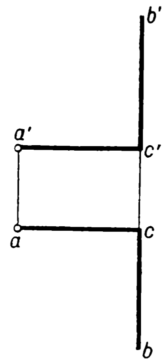

Fig. 91

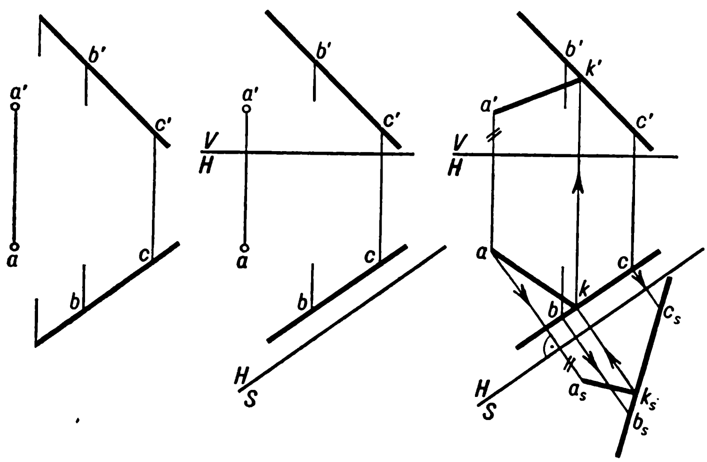

Fig. 92

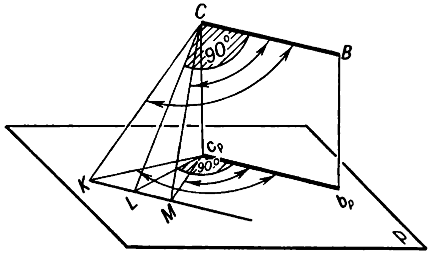

Fig. 93

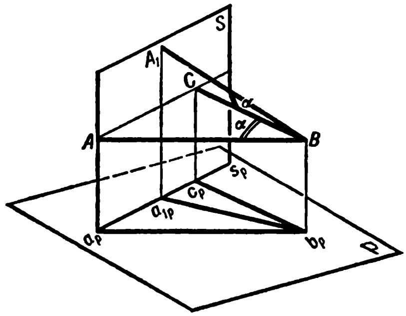

Fig. 94

May we consider that by having constructed the perpendicular AK to the line BC, the distance from A to BC is determined? No, we have only constructed the projections of the line segment AK neither of which determines the true length of the distance. If we have to determine the length of the segment AK, i.e. the distance from A to BC, the construction should be continued, say, by the method considered in Sec. 13.

5. *If a plane containing an obtuse or an acute angle is not perpendicular to a plane of projection and at least one of its sides is parallel to the projection plane, then the projection of the obtuse angle on this plane represents an obtuse angle, and the projection of the acute angle an acute angle.*

Suppose the line CB (Fig. 93) is parallel to the projection plane. Let us consider the obtuse angle KCB or the acute angle MCB, and draw in the plane of this angle a line CL perpendicular to CB. Since the angle LCB is a right one, its projection (the angle Lc_pb_p) represents also a right angle. This angle is enclosed inside the angle Kc_pb_p and contains the angle Mc_pb_p, consequently, the angle Kc_pb_p is obtuse, and Mc_pb_p is acute.

Thus, the projection of an angle represents an angle named as the projected angle itself (right, obtuse, or acute) if at least one side of the angle is parallel to the plane of projection.

In general, the projection of any angle may represent an acute, or a right, or an obtuse angle depending on the position of the angle relative to the projection plane.

6. *If both sides of any angle are parallel to the plane of projection, then its projection is equal by magnitude to the angle projected.*

This follows from the equality of angles with parallel and equally directed sides.

That is why it is easy to determine, for instance, the angle between the line AB (see Fig. 50) and the plane V, since this is the angle between the projection ab and the x-axis; analogously, the angle between CD and the plane H (Fig. 51) is determined as the angle between $c'd'$ and the x-axis, and the angle between EF (Fig. 52) and the plane V as the angle between $e''f''$ and the z-axis.

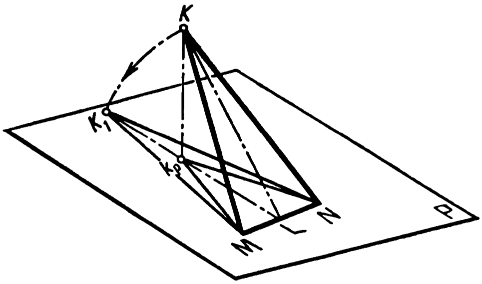

Fig. 95

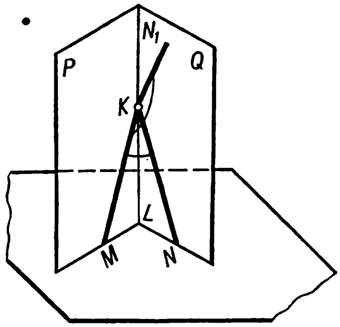

Fig. 96

For a right angle the equality of its projection to the angle itself is preserved also when only one side of the right angle is parallel to the projection plane.

But for an acute or for an obtuse angle one of whose sides is parallel to the projection plane the projection of the angle cannot be equal to the projected angle.

Moreover, the projection of an *acute* angle is less than the angle projected, and the projection of an *obtuse* angle is greater (by its magnitude) than the angle itself.

Let (Fig. 94) A_1BC be an acute angle and its side CB be parallel to the plane P; c_pb_p is parallel to CB. The plane S drawn through the point C perpendicular to CB is perpendicular to the plane P and intersects the latter along the line S_p passing through c_p perpendicular to c_pb_p. If we draw through the point B different straight lines at the same acute angle to CB, then all these lines will intersect the plane S at points whose projections will be located on the line S_p. Let us assume that the lines AB and A_1B form with the line CB equal angles: $\angle ABC = \angle A_1BC$. And if AB is parallel to the plane P, then $\angle a_pb_pc_p = \angle ABC$. If the side A_1B is not parallel to P, then the projection of the point A_1 is obtained on the line S_p nearer to c_p than the projection of the point A. Consequently, the projection of the angle A_1BC represents an angle *smaller* than the angle $a_pb_pc_p$, i.e. $\angle a_{1p}b_pc_p < \angle$ $\angle A_1BC$.

7. *If the sides of an angle are parallel to the projection plane or inclined to it at equal angles, then the bisection of the projection of the angle on this plane of projection corresponds to the bisection of the angle itself in space.*

8. *The bisection of an angle in space corresponds to the bisection of its projection only provided the sides of the angle form equal angles with the projection plane.*

9. *If the sides of an angle are inclined to the projection plane at equal angles, then its projection cannot be equal to the angle itself.*

This can be proved by bringing the angle MKN into coincidence with the plane P when rotating it about the line MN (Fig. 95). As is obvious from the drawing, the angle Mk_pN will turn out to be inside the angle MK_1N, and the vertices K_1 and k_p on a common perpendicular to MN.

10. *The projections of an acute and an obtuse angles may be equal to the angle projected not only under the condition of parallelism of the sides of the angle to the projection plane.*

It is seen from Fig. 96 that all the angles, for instance, the acute angle MKN and the obtuse angle MKN_1 whose sides are respectively contained in the projecting planes P and Q are projected into an angle equal to the angle MLN, these angles approaching 0° and 180°, respectively. Obviously, among these angles one may appear to be equal to its projection.

An example of constructing such an angle is given in Sec. 38.

QUESTIONS TO SECS. 13-15

1. How are right triangles constructed on the drawing for determining the length of a segment of an oblique line and its angles with the projection planes V and H?

2. What conditions must be satisfied by the angles between an oblique line and the projection planes V and H?

3. What property of parallel projecting refers to parallel lines?

4. Is it possible to find out whether two profile lines are parallel to each other given the drawing of these lines in the system V, H?

5. How are two intersecting lines represented in the system V, H?

6. How should the point of intersection of the projections of two skew lines be interpreted?

7. In what case is a right angle projected in the form of a right angle?

8. In what case is the projection of an obtuse or an acute angle is necessarily an angle named accordingly (obtuse or acute)?

9. Is it possible for the projection of an acute or of an obtuse angle one of whose sides is parallel to the projection plane to be equal to this angle in space?

10. In what case does the bisection of the angle obtained in projection correspond to such bisection of the angle in space?

11. Is it possible for an angle obtained as the projection of an angle in space to be equal to this angle if the sides of the latter form equal angles with the projection plane?

12. Is it possible for an acute or for an obtuse angle whose sides are not parallel to the projection plane to be equal to its projection on this plane?

CHAPTER 3

THE PLANE

Sec. 16. Ways of Specifying a Plane in the Drawing

The position of a plane in space may be determined by:

(1) three points not lying on one line; (2) a line and a point not lying on the line; (3) two intersecting lines; (4) two parallel lines.

Accordingly, a plane in the drawing may be specified by: (1) the projections of three points not lying on one line (Fig. 97); (2) the projections of a line and a point not lying on the line (Fig. 98); (3) the projections of two intersecting lines (Fig. 99); (4) the projections of two parallel lines (Fig. 100).

The specifications of a plane represented in Figs. 97-100 can be transformed into one another. For instance, drawing a line through the points A and B (Fig. 97), we obtain the specification of a plane represented in Fig. 98, wherefrom we can pass over to Fig. 100 by drawing through the point C a line parallel to the line AB. A plane may be specified on the drawing by the projections of any plane figure (a triangle, square, circle, etc.). Let a plane P be defined by the points A, B, and C (Fig. 101). Drawing straight lines through the like projections of these points, we get the projections of the triangle ABC. The point D taken on the line AB thus belongs to the plane P; drawing a line through the point D and another point *a fortiori* belonging to the plane P (for instance, through the point C), we get another line in the plane P.

Analogously, we may construct straight lines and, consequently, points belonging to a plane specified by any of the above-listed methods.

We shall see below that a *plane perpendicular to the projection plane may be specified by the straight line along which these planes intersect.*

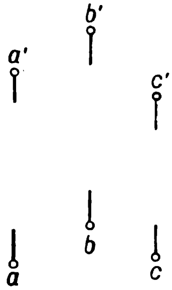

Fig. 97

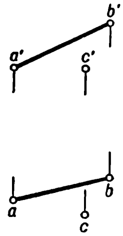

Fig. 98

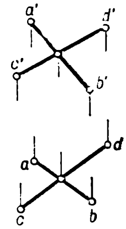

Fig. 99

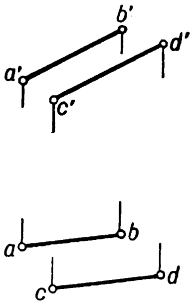

Fig. 100

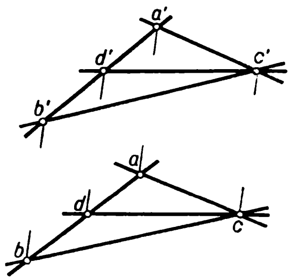

Fig. 101

Sec. 17. Constructing Traces of a Plane

A more obvious representation of a plane can be obtained by means of straight lines along which it intersects the projection planes. Fig. 102 gives an example of constructing such lines when a plane Q is specified by two intersecting lines AB and CB.

To construct the straight line along which the plane Q intersects the plane H it is sufficient to construct two points belonging both to the plane Q and to the plane H. Such points are the traces of the lines AB and CB on the plane H, i.e. the points of intersection of these lines with the plane H. We thus construct the projections of these traces and draw a line through the points m_1 and m_2 to obtain the horizontal projection of the line of intersection of the planes Q and H.

The line of intersection of the planes Q and V is defined by the vertical traces of the lines AB and CB.

Straight lines along which a plane intersects the projection planes are called the traces of this plane on the projection planes, or simply the *traces of the plane*.

Figure 103 represents a plane P which intersects the horizontal plane of projection along a straight line designated P_h and the vertical plane along a

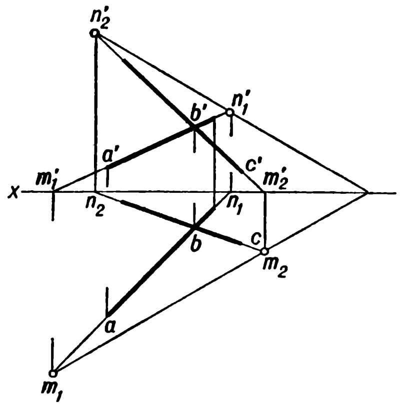

Fig. 102

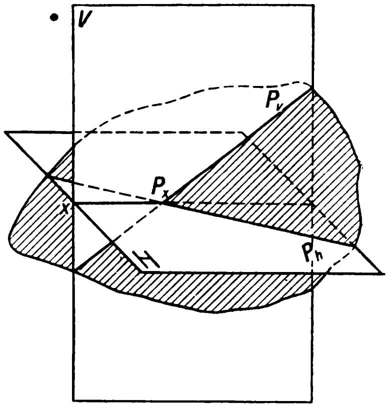

Fig. 103

straight line P_v. The line P_h is called the horizontal trace of the plane, and the line P_v its vertical trace.

If a plane intersects the projection axis, then a point of intersection of the traces of the plane is obtained on this axis. In Figure 103 the traces P_v and P_h intersect on the x-axis at a point designated P_x.

The trace of a plane on the projection plane merges with its projection on this plane. The trace P_h (Fig. 103) merges with its horizontal projection; the vertical projection of this plane is located on the axis of projection. The trace P_v merges with its vertical projection; the horizontal projection of this trace is situated on the axis of projection.

In the drawing a plane may be specified by the projections of its traces. We may confine ourselves to designating only the traces themselves (Fig. 104). Such a drawing is descriptive and is convenient in carrying out some constructions.

When constructing the traces of a plane the point of their intersection may be used for checking the accuracy of the construction: the traces must intersect at a point on the projection axis (see Fig. 102).

The angle between the traces in the drawing is not equal to the angle formed by the traces of a plane in space. Indeed, found at the intersection of the traces is the vertex of a trihedral angle two of whose faces coincide with the planes of projection (Fig. 103). But the sum of two plane angles of a trihedral angle exceeds the third plane angle. That is why the angle formed by the traces P_v and P_h in the drawing (Fig. 104) is always greater than the angle between these traces in space.

Considering a plane in the system V, H, W, we come to a conclusion that in the general case a plane intersects each of the axes of projection (in Fig. 105 the plane P intersects the x-, y-, and z-axes). Such plane is called an *oblique plane*. The trace P_w is termed the *profile trace of the plane*.

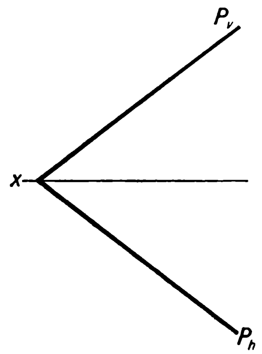

Fig. 104

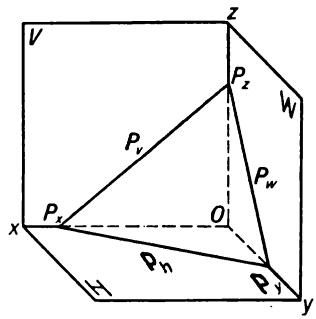

Fig. 105

Since the points P_x, P_y, and P_z lie on the x-, y-, and z-axes, respectively, then to construct the drawing of a plane in the system V, H, W it is sufficient to have the line segments OP_x, OP_y, and OP_z specified, i.e. to know the coordinates of the points P_x, P_y, and P_z in the system of x-, y-, and z-axes. In other words, it is sufficient to know only one coordinate for each of the points, since two other coordinates are equal to zero. For instance, to construct the point P_z we have to know only its z-coordinate, since the abscissa and ordinate of this point are zero.

Sec. 18. A Straight Line and a Point in the Plane. Principal Lines of a Plane

How do we construct in the drawing a straight line contained in a given plane? This construction is based on two statements known from geometry.

(1) *A straight line belongs to a plane if it passes through two points belonging to this plane.*

(2) *A straight line belongs to a plane if it passes through a point belonging to this plane and is parallel to a line contained in this plane or in a plane parallel to the given one.*

Suppose that the plane Q_1 (Fig. 106) is defined by two intersecting lines AB and CB, and the plane Q_2 by two parallel lines DE and FG. According to the first statement, a line intersecting the lines defining the plane is contained in this plane.

Whence it follows that if a plane is specified by its traces, then a *line belongs to the plane if its traces lie on like traces of the plane* (Fig. 107).

Suppose the plane P (Fig. 106) is defined by a point A and a line BC. According to the second statement, a line drawn through the point A and parallel to the line BC belongs to the plane P. Hence, a *line belongs to a plane if it is parallel to one of the traces of this plane and has a common point with the other trace* (Fig. 108).

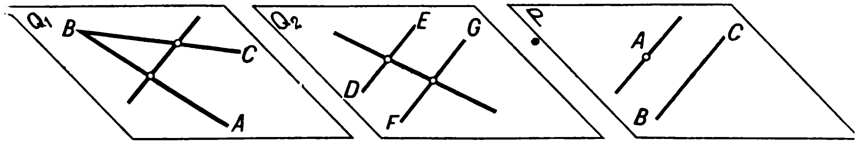

Fig. 106

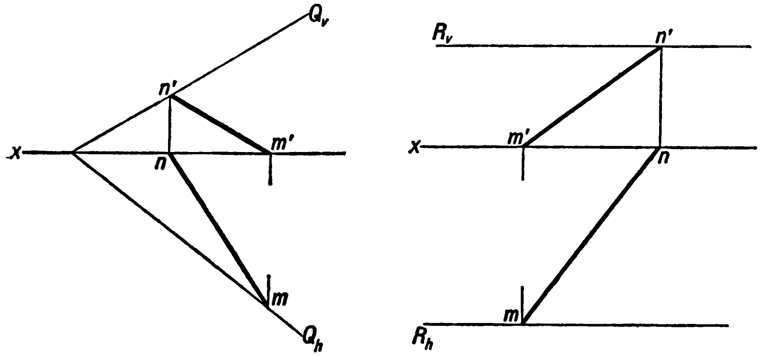

Fig. 107

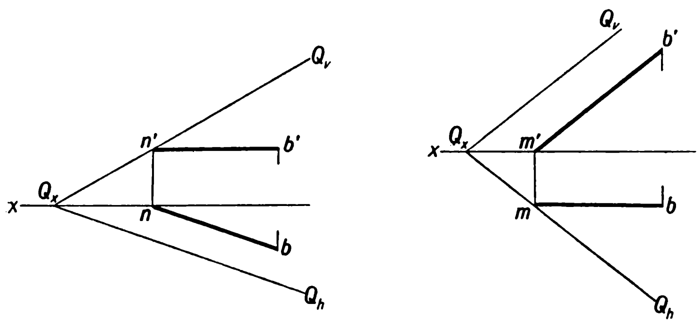

Fig. 108

The examples of constructions given in Figs. 107 and 108 should not be understood so that prior to constructing a line in a given plane we have necessarily to construct the traces of this plane. Of course, it is not required to.

For instance, in Fig. 109 a line *AM* is constructed in a plane specified by a point *A* and a line passing through the point *L*. Let us assume that the line *AM* must be parallel to the plane *H*. We begin with drawing the projection $a'm'$ perpendicular to the line of recall $a'a$. Using the point m', we find the

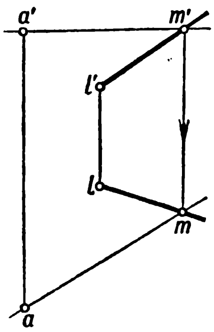

Fig. 109

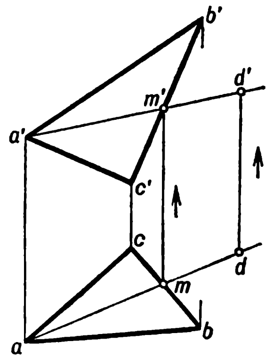

Fig. 110

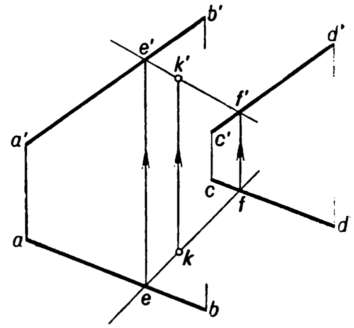

Fig. 111

point *m* and then draw the projection *am*. The line *AM* satisfies the initial condition: it is parallel to the plane *H* and lies in the given plane, since it passes through two points (*A* and *M*) *a fortiori* belonging to this plane.

How to construct in the drawing a point contained in a given plane? Prior to doing this, *we construct a line lying in the given plane and take a point on this line.*

For example, it is required to find the vertical projection of the point *D* if its horizontal projection *d* is given, and it is known that the point *D* must lie in the plane defined by the triangle *ABC* (Fig. 110).

First of all we construct the horizontal projection of a straight line so that the point *D* might appear on this line, and the latter would be contained in the given plane. To this end we draw a line through the points *a* and *d* and mark the point *m* at which the line *ad* intersects the line segment *bc*. By constructing the vertical projection *m'* on *b'c'*, we get the line *AM* situated in the given plane: this line passes through the points *A* and *M* of which the first one *a fortiori* belongs to the given plane, the second being constructed in it.

The required vertical projection *d'* of the point *D* must lie on the vertical projection of the line *AM*.

Another example is given in Fig. 111. In the plane *Q* specified by parallel lines *AB* and *CD* there must be a point *K* for which only the horizontal projection (point *k*) is given. Through the point *k* there drawn a line taken as the horizontal projection of the line in a given plane. Using the points *e* and *f*, we construct *e'* on *a'b'* and *f'* on *c'd'*. The line *EF* thus constructed belongs to the plane *Q*, since it passes through the points *E* and *F a fortiori* belonging to the plane. If a point *k'* is taken on *e'f'*, then the point *K* will turn out to be contained in the plane *Q*.

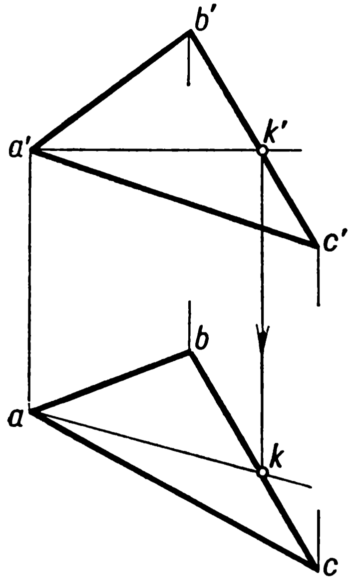

Fig. 112

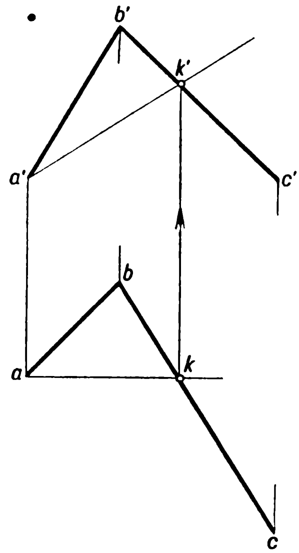

Fig. 113

Of the straight lines that may be situated in a given plane, of special importance are the following lines: *H principal lines* (also called *H parallels* or *horizontal lines*), *V principal lines* (also called *V parallels* or *vertical lines*)*, and the steepest lines, i.e. the *lines of maximum inclination to the projection planes.* The line of maximum inclination to the plane *H* will be called the *slope line of a plane***.

Horizontal lines are lines lying in a given plane and parallel to the horizontal plane of projection.

Let us construct a horizontal line of the plane specified by the triangle *ABC*. It is required to draw the horizontal line through the vertex *A* (Fig. 112).

Since the horizontal line of a plane is a straight line parallel to the plane *H*, we obtain its vertical projection by drawing $a'k'$ perpendicular to $a'a$. To get the horizontal projection of this horizontal line we construct the point *k* and draw a straight line through the points *a* and *k*.

The line *AK* thus constructed is really the horizontal line of the given plane: it lies in the plane, since it passes through two points which *a fortiori* belong to this plane, and is parallel to the *H* plane of projection.

---

*Along with horizontal and vertical lines of a plane, we may also consider its profile lines, i.e. lines lying in a given plane and parallel to the *W* plane.

**The slope line of a plane is often called "the line of the greatest slope", but the notion 'slope' with respect to a plane does not require the attribute 'greatest'.

Let us now consider the construction of a horizontal line of a plane specified by its traces.

The horizontal trace of a plane is one of its H parallels (a "zero" parallel). Therefore the construction of an H parallel of a plane is reduced to drawing in this plane a line parallel to the horizontal trace of the plane (Fig. 108, left). The horizontal projection of a horizontal line is parallel to the horizontal trace of a plane; the vertical projection of a horizontal line is parallel to the axis of projection.

Vertical lines are lines lying in a given plane and parallel to the vertical plane of projection.

An example of constructing a vertical line in a plane is given in Fig. 113. The construction is carried out analogously to that of a horizontal line (see Fig. 112).

Let a vertical line pass through the point A (Fig. 113). We begin the construction with drawing the horizontal projection of the vertical line, i.e. the line ak, since the direction of this projection is known: ak is perpendicular to $a'a$. We then construct the vertical projection of the vertical line, that is the line $a'k'$.

The straight line thus constructed is really a vertical line of the given plane: this line is contained in the plane, since it passes through two points belonging to this plane, and is parallel to the V plane.

Let us now construct a vertical line in a plane specified by its traces. Examining Fig. 108 (right) in which a plane Q and a straight line MB are represented, we find out that this line is a vertical line of the given plane. Indeed, it is parallel to the vertical trace of the plane (i.e. to its "zero" vertical line). The horizontal projection of the vertical line is parallel to the x-axis, its vertical projection being parallel to the vertical trace of the plane.

The steepest lines of a plane with respect to the planes H, V, and W are lines lying in this plane and perpendicular to its horizontal lines, or to its vertical lines, or to its profile lines. In the first case the inclination to the H plane is determined, in the second case to the V plane, and in the third to the W plane. To construct the steepest lines of the plane we may use, of course, its traces.

As it was said above, the steepest line with respect to the H plane is called the *slope line of a plane.*

According to the rules for projecting a right angle (see Sec. 15), the horizontal projection of the slope line of a plane is perpendicular to the horizontal projection of a horizontal line of this plane or to its horizontal trace. The vertical projection of the slope line is constructed after its horizontal projection and may occupy various positions depending on how the plane is specified. Fig. 114 illustrates the slope line of the plane Q: BK is perpendicular to Q_h. Since bK is also perpendicular to Q_h, the angle BKb is a plane angle of the dihedral angle formed by the planes Q and H. Consequently, the *slope line of a plane may serve for determining the angle of inclination of this plane to the H plane of projection.*

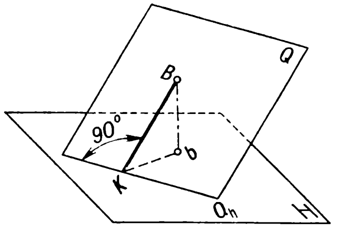

Fig. 114

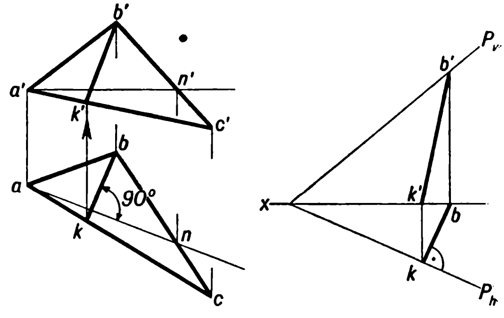

Fig. 115

Analogously, the steepest line of a plane with respect to the *V* plane serves for determining the angle between this plane and the *V* plane of projection, and the steepest line with respect to the *W* plane for determining the angle with the *W* plane.

Figure 115 shows the slope lines constructed in given planes. The angle between the planes *P* and *H* is represented by the projections of the line segment *BK*. The magnitude of this angle can be determined by constructing a right triangle using the projections $b'k'$ and bk.

Obviously, the steepest line of a plane determines the position of this plane. For instance, if the slope line *KB* is specified (Fig. 115), then, drawing a horizontal line *AN* perpendicular to it, or given the *x*-axis of projection and drawing P_h perpendicular to *kb*, we completely determine the plane for which *KB* is the slope line.

The above considered principal lines of a plane, mainly horizontal and vertical lines, are often used in various constructions and when solving problems. This is because these lines are easily constructed and therefore they are convenient to be used as auxiliary lines.

In Figure 116 we are given the horizontal projection *k* of the point *K*. It is required to find the vertical projection k' of the point *K* which must be in the plane specified by two parallel lines drawn from the points *A* and *B*.

First we have to draw a straight line passing through the point *K* and lying in the given plane. Here it is convenient to choose a vertical line *MN*: its horizontal projection is drawn through the given projection *k*. Then we construct points m' and n' determining the vertical projection of the vertical line.

The required projection k' must be situated on the line $m'n'$.

In Figure 117 (left) given the vertical projection a' of point *A* belonging to the plane *P*, its horizontal projection *a* is found. The appropriate construction is accomplished with the aid of a horizontal line *EK*.

In Figure 117 (right) an analogous problem is solved with the aid of a vertical line *MN*.

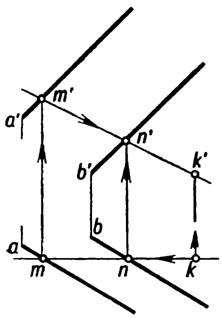

Fig. 116

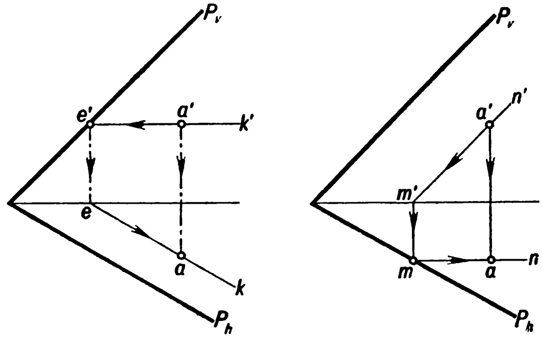

Fig. 117

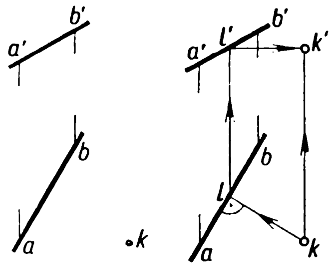

Fig. 118

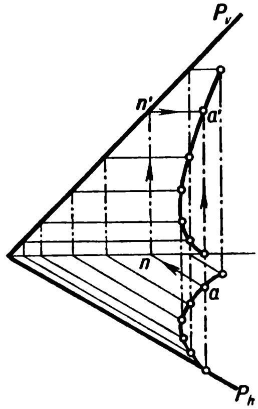

Fig. 119

One more example of constructing the missing projection of a point belonging to some plane is given in Fig. 118. Given: the slope line *(AB)* of a plane and the horizontal projection (*k*) of a point (left); required: to construct the vertical projection of the point. Figure 118 (right) shows the construction: through the point *k* we draw (perpendicular to *ab*) the horizontal projection of the horizontal line on which the point *K* must lie. Using the point *l'* we find the vertical projection of this horizontal line and on it the required projection *k'*.

Figure 119 gives an example of constructing a second projection of a plane curve, using the known horizontal projection of this curve and a plane *P* in which this curve is contained. Taking a number of points on the horizontal projection of the curve and using horizontal lines, we find the points required for constructing the vertical projection of the curve.

As usual, arrows indicate the successive steps in finding the vertical projection *a'* with the horizontal projection *a* known.

QUESTIONS TO SECS. 16-18

1. How is a plane specified in the drawing?
2. What is the trace of a plane on the plane of projection?
3. What is the location of the vertical projection of the horizontal trace and the horizontal projection of the vertical trace of a plane?
4. How is it determined from the drawing whether a straight line belongs to a given plane?
5. How is a point belonging to a given plane constructed in the drawing?
6. What is a vertical line, a horizontal line, and the steepest line of a plane?
7. Can the slope line of a plane serve for determining the angle of inclination of this plane to the H plane of projection?
8. Does a straight line define a plane for which this line is the slope line?

Sec. 19. Various Positions of a Plane Relative to the Projection Planes

The following positions of a plane relative to the projection planes V, H, W are possible: (1) *a plane is perpendicular to none of the projection planes,* (2) *a plane is perpendicular to one of them,* (3) *a plane is perpendicular to two planes of projection.*

Planes grouped under (2) and (3) are termed 'projecting planes'.

1. *A plane perpendicular to none of the projection planes is an oblique plane* (see Fig. 105).

Let us, for instance, consider the plane represented in Fig. 112.

This plane is perpendicular to none of the projection planes. That it is perpendicular neither to V, nor to H is confirmed by the form of the projections $a'b'c'$ and abc: if the plane defined by the triangle ABC were perpendicular at least to H, then the projection abc would be a segment of a straight line (Fig. 120).

Thus, the plane under consideration is perpendicular neither to V, nor to H. Then, maybe it is perpendicular to W? No, the horizontal line AK of this plane is not perpendicular to W (cf. Fig. 54 showing a straight line perpendicular to W), and, consequently, the plane ABC is not perpendicular to W.

Hence, Fig. 112 gives an example of specifying an oblique plane in the system V, H.

More examples of specifying an oblique plane are given in Figs. 109, 110, 111, 113, 116, as also in Figs. 102, 104, 107 (left), 108, 115 (right), 117, and 119 in which planes are represented by their traces. *An oblique plane* (see Fig. 105) *intersects each of the axes x, y, z. The traces of an oblique plane are never perpendicular to these axes of projection.*

If the traces P_h and P_v of an oblique plane form equal angles with the x-axis, then it means that the angles between the plane P and the planes H

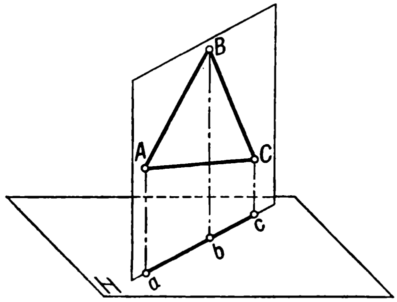

Fig. 120

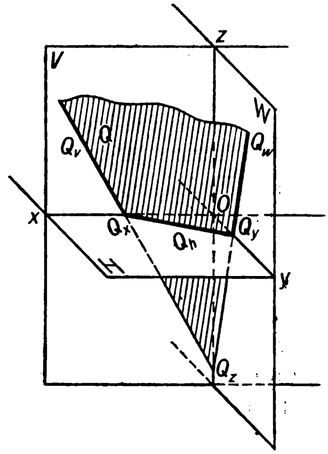

Fig. 121

and V are equal to each other. Indeed, if the plane angles of a trihedral angle are equal to each other, then the dihedral angles opposite them are also equal. The angles formed by the traces P_h and P_v with the x-axis (see Fig. 105) represent the plane angles opposite which there respectively situated the dihedral angles formed by the plane P with the planes V and H.

For an oblique plane to be inclined to the planes H, V, and W at equal angles, it is necessary that $OP_x = OP_y = OP_z$ (Fig. 105), i.e. the traces must form with the projection axes angles equal to 45°.

Considering an oblique plane in space within the limits of the first quadrant or the first octant, we note that the angle between the horizontal and vertical traces may be acute (as in Fig. 105) or obtuse (as in Fig. 121).

The plane Q depicted in Fig. 121 passes through all the octants except for the sixth.

If the drawing of an oblique plane is to be constructed with the aid of the coordinates of the points of intersection of its traces, then, obviously, the following must be given: the positive abscissas and ordinates of the points Q_x and Q_y, and negative z-coordinate of the point Q_z.

Figure 122 illustrates a particular case of an oblique plane: its traces P_h and P_v lie on one line in the drawing. With Fig. 15 in mind, we note that the traces P_h and P_v form equal angles with the x-axis not only in the drawing, but in space as well. As is shown in Fig. 122 (right), from the congruence of the right triangles k_0kP_x and $k'kP_x$ it follows that the angle k_0P_xk is equal to the angle kP_xk', i.e. the traces P_v and P_h form equal angles with the x-axis.

Hence, the plane P forms equal angles with the planes H and V. Its portion situated in the first quadrant contains the true size of the angle between P_h and P_v (in our example it is obtuse).

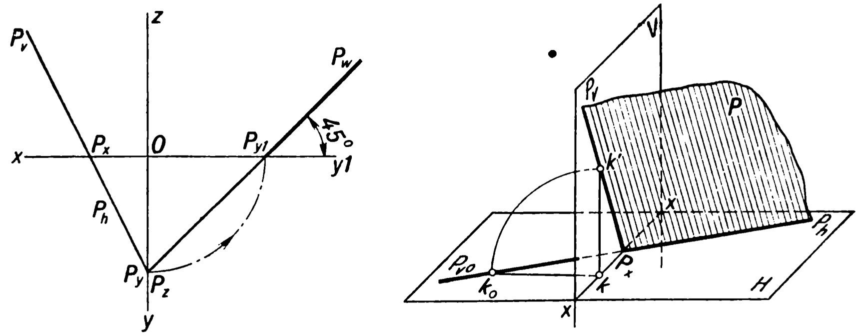

Fig. 122

Figure 122 shows also the construction of the third trace of a plane (P_w) using its two traces P_h and P_v. Since the traces P_h and P_v lie on one line, the point P_z merges with the point P_y and, consequently, the point Py_1 turns out to be equidistantwi th the point P_z from the point O. Therefore the trace P_w is inclined to the y-axis (and to the z-axis) at an angle of 45°. Such an inclination of the profile trace will occur in all cases of constructing a plane whose horizontal and vertical traces lie on one line intersecting the x-axis at an acute angle.

This plane passes through a perpendicular to the x-axis which forms an angle of 45° with the V (or H) plane. And since this perpendicular is perpendicular to the bisector plane of the dihedral angles adjacent to the angle VH, the plane under consideration may be defined as a plane perpendicular to the bisector plane of the second and fourth quadrants.

2. If a plane is perpendicular only to one projection plane, then three particular cases are possible.

(a) *A plane is perpendicular to the horizontal plane of projection. These are called horizontal projecting planes.*

An example is given in Fig. 123: a plane is specified by the projections of a triangle ABC. Its horizontal projection represents a segment of a straight line. The angle β is equal to the angle between the given and the V plane.

Figure 124 illustrates an example of representation of a horizontal projecting plane by its traces: along with the pictorial representation (left), we are given (in the middle) a drawing in the system V, H with the x-axis and traces S_v and S_h indicated, and (right) a drawing without indicating the x-axis and, hence, the S_v trace.

The vertical trace is perpendicular to the H plane and to the x-axis of projection, the horizontal trace forming any angle with the projection axis. This angle serves as a plane angle of the dihedral angle between the horizontal projecting plane and the V plane.

The angle between S_h and S_v, as also between S_h and S_w in space is equal to 90°.

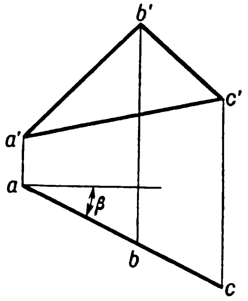

Fig. 123

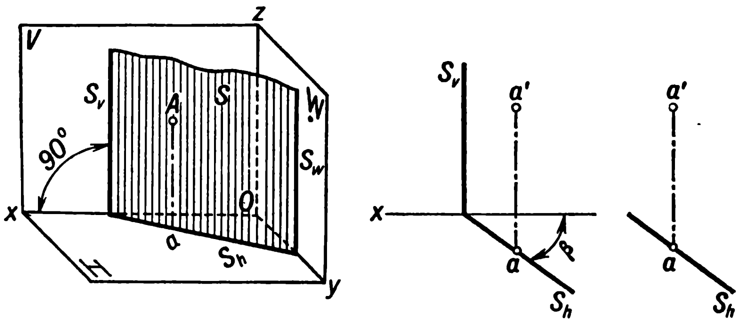

Fig. 124

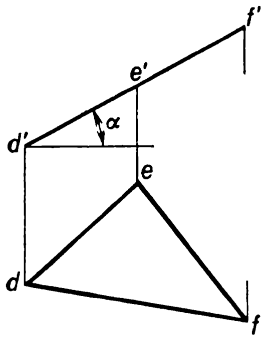

Fig. 125

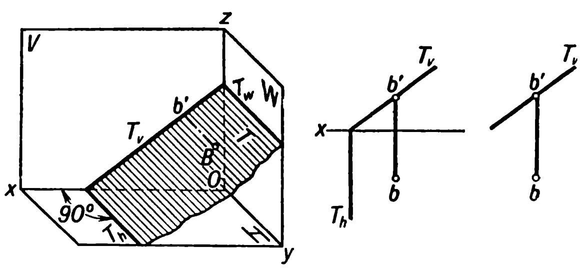

Fig. 126

If a horizontal projecting plane contains a point, then the horizontal projection of this point must lie on the horizontal projection of the plane. It refers also to any system of points contained in a horizontal projecting plane irrespective of whether it is a straight line, a plane curve, or a figure.

The trace S_h may be considered as the horizontal projection of the plane.

(b) *A plane is perpendicular to the vertical plane of projection. These are called vertical projecting planes.* An example is given in Fig. 125: a plane is specified by the projections of a triangle *DEF*. The vertical projection represents a segment of a straight line. The angle α is equal to the angle between *DEF* and the *H* plane.

Figure 126 gives a pictorial representation (left), a drawing in the system *V*, *H* with the projection axis shown (middle), and a drawing without showing the axis of projection (right). The horizontal trace is perpendicular both to the *V* plane and to the projection axis, the vertical trace forming any angle with the projection axis. This angle serves as a plane angle of the dihedral angle between the vertical projecting plane and the *H* plane.

The angle between T_v and T_h in space is equal to 90°.

If a vertical projecting plane contains a point, then the vertical projection of this point must lie on the vertical trace of the plane. It is true for any system of points. The trace T_v (Fig. 126) may be considered as the vertical projection of the plane *T*.

(c) *A plane is perpendicular to the profile plane of projection. Such planes are called profile projecting planes.*

Figure 127 offers an example of a profile projecting plane which is specified by the projections of a triangle *ABC*. The horizontal of this plane is perpendicular to the *W* plane: the projections $a'd'$ and ad are parallel to each other. This testifies to the fact that we deal with a profile projecting plane but not with an oblique plane (cf. Fig. 112).

The profile projection of the triangle *ABC* represents a segment of a straight line. The angle α between this segment and the line of recall $c'c''$ is equal to the angle of inclination of the triangle to the *H* plane, and the angle of inclination of the plane containing the triangle to the *V* plane is equal to $90°-\alpha$.

Figure 128 illustrates an example of representing a profile projecting plane by its traces.

The horizontal and vertical traces of this plane are parallel to the x-axis and, consequently, are parallel to each other.

The plane represented in Fig. 107 (right) is also a profile projecting plane.

A plane perpendicular to one of the projection planes (i.e. a horizontal-, vertical-, or profile-projecting plane) may, in particular, pass through the axis of projection. Such plane is called an axial plane.

Let us, for instance, consider the axial profile projecting plane represented in Fig. 129. Its traces R_v and R_h merge with the x-axis; in this case it is necessary to have its third trace R_w or at least the position of a point belonging to this plane and not lying on the x-axis.

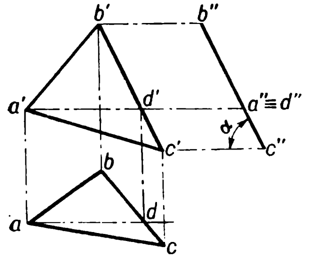

Fig. 127

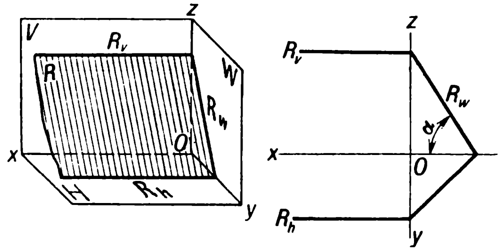

Fig. 128

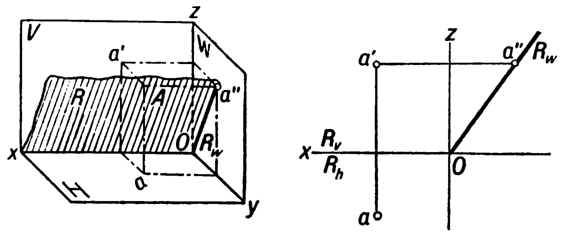

Fig. 129

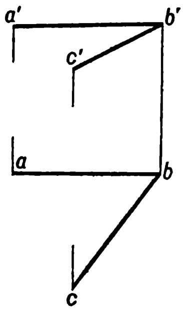

Fig. 130

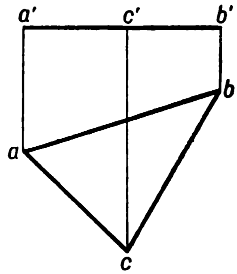

Fig. 131

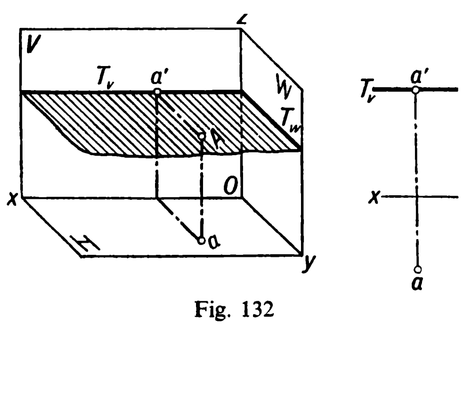

Fig. 132

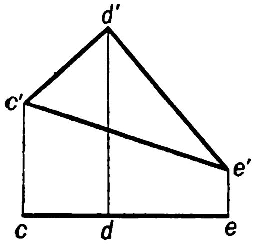

Fig. 133

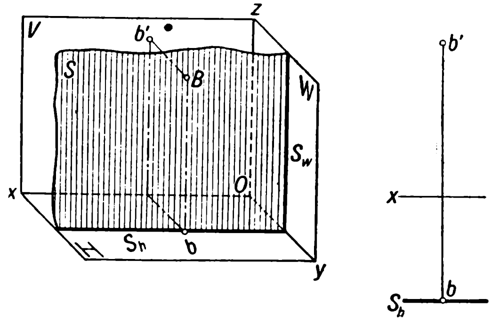

Fig. 134

An axial plane may be a *bisector* one. This means that an axial plane bisects the dihedral angle formed by the projection planes.

How can we represent a profile projecting plane in the drawing without the projection axes? In the way shown in Fig. 127. Another example is given in Fig. 130: the plane is specified by two intersecting lines one of which (AB) is perpendicular to the W plane, the other occupying an arbitrary position.

3. If a plane is perpendicular to two planes of projection, then also three cases of particular positions are possible:

(a) *A plane is perpendicular to the planes V and W, i.e. parallel to the H plane. These are called horizontal planes.*

Figure 131 illustrates an example of a horizontal plane specified by the projections of a triangle ABC. In Figure 132 (right) a horizontal plane in the system V, H is represented by its vertical trace. This trace (T_v) may be considered as the vertical projection of the plane.

(b) *A plane is perpendicular to the planes H and W, i.e. parallel to the V plane. Such planes are called vertical planes.*

Figure 133 is an example of a vertical plane specified by the projections of a triangle CDE.

Figure 134 (right) gives an example of representing a vertical plane in the system V, H by its trace S_h which may be considered as the projection of this plane on the H plane.

(c) *A plane is perpendicular to the planes H and V, i.e. parallel to the plane W. These are profile planes.*

An example of representation of such a plane in the system V, W is given in Fig. 135: the plane is specified by the projections of a triangle EFG.

Another example is given in Fig. 136. Here the plane is represented in the system V, H by its traces. Either of them may be considered as the projection of the plane P on the corresponding plane of projection. A profile plane combines in itself the properties of both a vertical- and a horizontal-projecting planes.

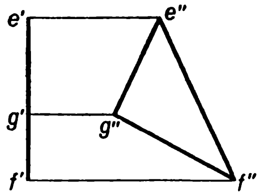

Fig. 135

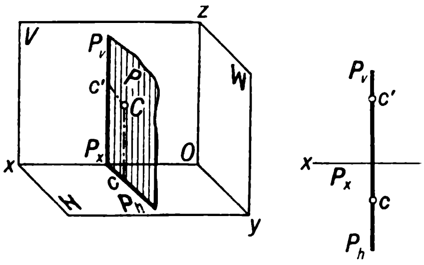

Fig. 136

QUESTIONS TO SEC. 19

1. How is an oblique plane arranged in the system *V*, *H*, *W*? The same question about planes called projecting.

2. What is a vertical projecting plane, a horizontal projecting plane, a profile projecting plane?

3. How is it possible to find out whether a plane specified in the system *V*, *H* by intersecting or parallel lines is an oblique plane or a profile projecting plane?

4. What do the horizontal projections of a horizontal projecting plane and a vertical plane represent?

5. The same question with reference to the vertical projections of a vertical projecting plane and a horizontal plane.

6. Where is the horizontal projection of any system of points contained in a horizontal projecting plane or a vertical plane situated?

7. Where is the vertical projection of any system of points contained in a horizontal plane or a vertical projecting plane situated?

8. What is the value of the angle in space between the vertical and horizontal traces of a horizontal- and a vertical-projecting planes?

Sec. 20. Drawing a Projecting Plane Through a Straight Line

Below we shall come across a necessity to draw a projecting plane through a straight line according to some condition. Any of such planes can be drawn through an oblique line. Examples are given in Fig. 137. Through a straight line specified in the system *V*, *H* and passing through the point *K* there drawn the following planes: a vertical projecting plane represented by its vertical trace T_v; a horizontal projecting plane represented by its horizontal trace S_h; and a profile projecting plane defined, besides the given line *AK*, also by a straight line *AB* perpendicular to the *W* plane.

In Figure 138 the planes drawn through a given straight line are represented by traces. The position of the *x*-axis is either specified, or may be chosen.

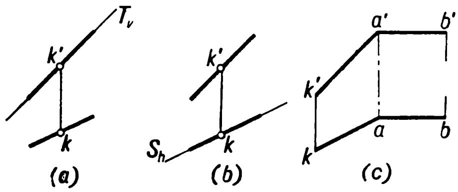

Fig. 137

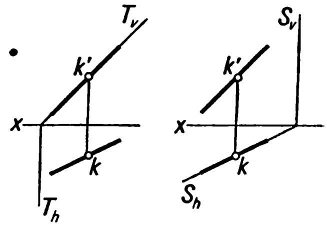

Fig. 138

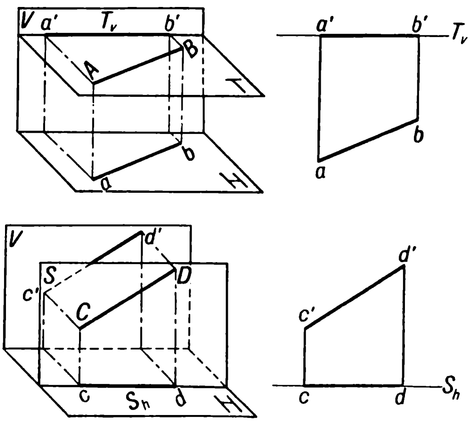

Fig. 139

But *no vertical, horizontal, or profile planes can be drawn through an oblique line.* Such planes can be drawn through lines situated correspondingly: a horizontal plane through a horizontal line, a vertical plane through a vertical line, a profile plane through a profile line. A horizontal plane T passing through a horizontal line AB, and a vertical plane S passing through a vertical line CD are shown in Fig. 139.

Sec. 21. Constructing the Projections of Plane Figures

The construction of projections of plane figures (i.e. of figures lying entirely in one plane—such as, a square, a circle, an ellipse, etc.) is reduced to constructing the projections of a number of points, segments of straight lines and curves which form the contours of projections of figures. Knowing the coordinates of the vertices of a triangle, we can construct the projections of these points, then the projections of the sides, and thus the projections of a figure.

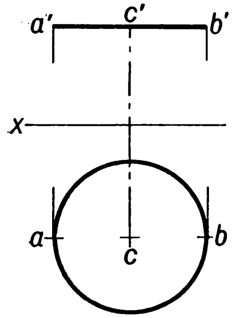

Fig. 140

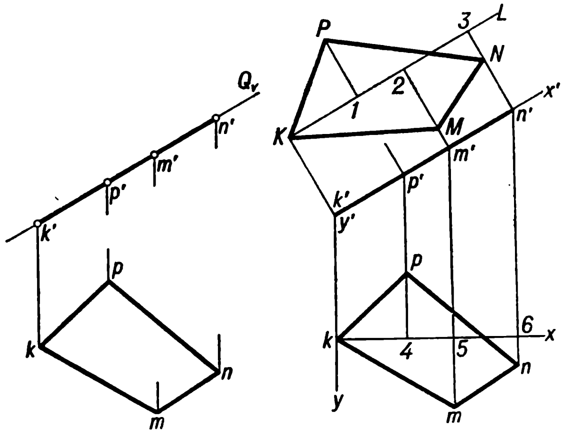

Fig. 141

We have already come across the drawings containing the projections of a triangle (for instance, Figs. 110, 112, and others). Comparing Figs. 110 and 112, we notice that in Fig. 110 one of the projections, say the vertical, represents the "front" side of the triangle, and the horizontal projection its "rear" side; whereas in Fig. 112 either of the projections represents the triangle from one and the same side. The order in which the vertices are traversed may serve as a test here: in Fig. 110 they are traversed clockwise (moving from a' to c') for the vertical projection, and anticlockwise for the horizontal projection; in Fig. 112 the vertices are traversed in one and the same direction (clockwise) in both projections.

In the general case the projections of a polygon in the system V, H, W represent also polygons with the same number of sides, the plane containing this polygon being an oblique plane. But if in the system V, H either of the projections of, say, a triangle represents a triangle, then its plane may turn out to be an oblique plane or a profile projecting plane: Fig. 112 illustrates an oblique plane, while Fig. 127 a profile projecting plane. As it was said in the explanation to Fig. 127, a horizontal (or vertical) line determines the situation in this case: if its projections on V and H are parallel to each other, then the plane is profile projecting (Fig. 127), but if they are not parallel, then we have an oblique plane (for example, Figs. 112 and 115, left).

If the projection of a polygon on V or on H represents a segment of a straight line, then the plane containing this polygon is perpendicular to V or to H, respectively. For instance, in Fig. 123 the triangle is contained in a horizontal projecting plane, and in Fig. 125 in a vertical projecting plane.

A figure placed parallel to a projection plane is projected on it without distortion (i.e. true size). For instance, all elements of the triangle CDE represented in Fig. 133 are projected on the V plane without distortion; the circle shown in Fig. 140 is projected on the H plane also without distortion.

But if the plane containing a figure is not parallel to a projection plane, then to determine the true shape of this figure we apply the methods discussed below (in Chapter 5). Of course, we could now, without knowing these methods yet, construct, for instance, the true size of the triangle represented in Fig. 112 by determining the true length of each of its sides as the length of a line segment (see Sec. 13), and constructing then a triangle from the line segments thus found. At the same time we would determine the angles of the given triangle. They usually proceed so, for instance, when constructing the development of the lateral surface of a pyramid, prism, etc. (see Sec. 44 below). If a polygon is contained in a projecting plane, we can obtain its true size proceeding as in Fig. 141.

Suppose it is required to determine the true shape of the quadrilateral *KPNM* contained in a vertical projecting plane *Q*. Then, as is shown in Fig. 141 (right), we may take two axes of rectangular coordinates in the plane of the figure with the origin, say, at *K*: the axis of abscissas ($k'x'$, kx) parallel to the *V* plane, the axis of ordinates perpendicular to the *V* plane (the projections of this axis are $k'y'$ and ky); draw a straight line *KL* (parallel, say, to $k'x'$), and lay off on it: $K1 = k'p'$, $K2 = k'm'$, $K3 = k'n'$. We complete the construction by laying off the line segments $P1 = p4$, $M2 = m5$, and $N3 = n6$ on the perpendiculars to the line *KL* at points *1*, *2*, and *3*. The quadrilateral *KMNP* thus constructed represents the true form of the given one.

When solving many problems determination of the position occupied by a plane figure relative to the projection planes is of essential importance. As an example, let us consider the problem of constructing the four remarkable points of the triangle.

Since to the bisection of a line segment in space there corresponds a similar bisection of the projections of this segment (see Sec. 12), the point of intersection of the medians* can be directly constructed in the drawing in all cases. It is sufficient (Fig. 142) to draw the medians on either of the projections of the triangle and the median point will be determined. In doing so we may confine ourselves to constructing both projections only of one of the medians (for instance, ad and $a'd'$) and one projection of a second median (for instance, $b'e'$); the intersection of $a'd'$ and $b'e'$ yields m'. Using this point we find the point m on ad.

We could make use of another method: to construct only one of the medians, and to find the point *M* on it taking advantage of the property of this point known from geometry: it divides each median in the ratio 2 : 1.

The construction of the point of intersection of the three altitudes of a triangle (i.e. the orthocentre of the triangle) *and of the point of intersection of the perpendiculars to the sides of a triangle drawn through their midpoints* (called the circumcentre) *is associated with drawing mutually perpendicular lines.*

*The point of intersection of the three medians is the centroid of the triangle.

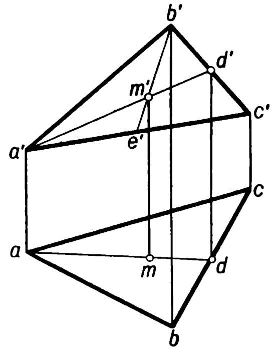

Fig. 142

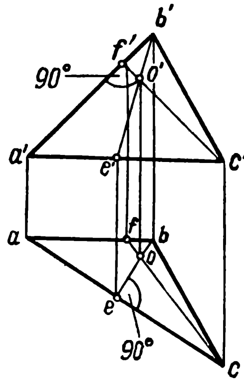

Fig. 143

In Section 15 we mentioned the conditions under which mutually perpendicular segments in space have also mutually perpendicular segments as their projections. If the plane of a triangle is parallel to a projection plane (for instance, the triangle *CDE* in Fig. 133), then dropping perpendiculars from the points *c'*, *d'*, and *e'* to the opposite sides, we obtain the projections of the altitudes of the triangle. But we cannot proceed in the same way with the triangle contained in an oblique plane.

In a particular case, when one of the sides of a triangle is parallel to the *H* plane, and the other is parallel to the *V* plane (Fig. 143), drawing *c'f'* perpendicular to *a'b'* and *be* perpendicular to *ac* we get in space *CF* perpendicular to *AB* and *BE* perpendicular to *AC*. The point of intersection of the altitudes is thus constructed without using any special methods.

But in the general case to draw perpendicular lines in the projection drawing we have to resort to special methods which will be discussed below.

The point of intersection of the bisectors (the incentre of the triangle) can also be constructed directly only in some particular cases. This is because the bisection of the projection of an angle corresponds to its bisection in space only if the sides of the given angle are equally inclined to the projection plane on which the projection of the angle is bisected (see Sec. 15)

When constructing the projections of a polygon pay attention that the condition of location of all points of a given figure in one plane is not violated.

Given in Fig. 144 (left) are a complete horizontal projection of a pentagon *ABCDE* and the vertical projections of only three of its vertices: *a*, *b'*, and *e'*. The right-hand drawing shows how the vertical projections of the two missing vertices (*c'* and *d'*) are constructed. For the points *C* and *D* to lie in the plane defined by three points (*A*, *B*, and *E*) it is necessary that they are situated on straight lines contained in this plane. Such lines are the diagonals *AC*, *AD*, and *BE* whose horizontal projections can be constructed. On the vertical projection of the pentagon we can draw only *b'e'*. But in the plane

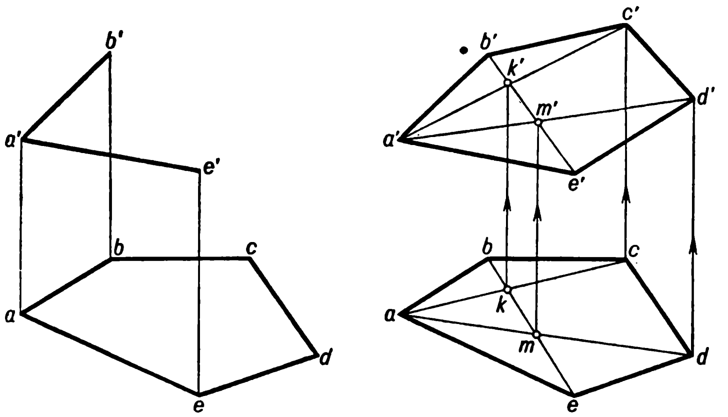

Fig. 144

of the pentagon there lie the points (K and M) of intersection of the diagonals whose horizontal projections (k and m) are available, and the vertical projections are obtained at once, since they must lie on $b'e'$. The vertical projections $a'k'$ and $a'm'$ of the two remaining diagonals are constructed by two points. The points c' and d' must lie on these diagonals, respectively. They are determined with the aid of their horizontal projections.

A circle whose plane is parallel to a projection plane is projected on this plane true size (see Fig. 140 in which a circle is taken in the horizontal plane). If the plane containing the circle is perpendicular to a plane of projection, then the circle is projected on this plane in the form of a line segment whose length is equal to the diameter of the circle.

But if a circle is situated in a plane which forms an acute angle α with the plane of projection, then the circle is projected into a figure called the ellipse.

Ellipse is also the name for the curve which bounds the figure of ellipse. Thus, the figure of ellipse is the projection of a circle, whereas the ellipse as a curve is the projection of its circumference. Henceforth the term 'ellipse' will mean the projection of the circumference of a circle.

The ellipse belongs to the curves called second-order or quadric curves (or simply quadrics). The equations of such curves in Cartesian coordinates represent equations of the second degree. A quadric curve intersects a straight line at two points. Below we shall come across the parabola and hyperbola which are also quadric curves.

The ellipse may be considered as a "compressed" circle. This is illustrated in Fig. 145 (left). Suppose a line segment OB_1 whose length is b is laid off on the radius OB, and $b < a$ (i.e. b is less than the radius of the circle). If we now take a point K on the circle and, drawing a perpendicular from K to A_1A_2, mark a point K_1 on KM so that $MK_1 : MK = b : a$, then the point K_1 will belong to the ellipse. Proceeding in this way we can transform any point of the circle into a point of the ellipse preserving one and the same

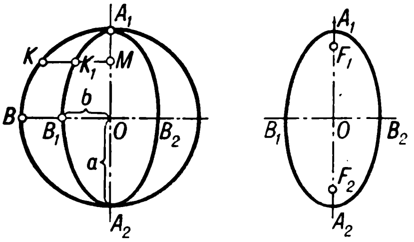

Fig. 145

ratio $b : a$. The circle thus becomes uniformly compressed; the line into which the circle is transformed is an ellipse. The ratio $b : a$ is called the compression coefficient. As b approaches a, the ellipse keeps expanding until it becomes a circle (when $b = a$).

When dealing with the ellipse the following should be remembered (from the high-school course of drawing):

(1) the line segment $A_1A_2 = 2a$ is called the *major axis* of the ellipse;
(2) the line segment $B_1B_2 = 2b$ is called the *minor axis* of the ellipse;
(3) the major and minor axes are mutually perpendicular;
(4) the point of intersection of the axes is called the *centre* of the ellipse;
(5) a line segment joining two points of the ellipse and passing through the centre of the ellipse is termed the *diameter* of the ellipse;
(6) points A_1, A_2, B_1, and B_2 are called the *vertices* of the ellipse;
(7) the ellipse is symmetric with respect to its axes and its centre;
(8) the ellipse is the locus of points the sum of whose distances from two given points F_1 and F_2 (Fig. 145, right) is constant and is equal to $2a$ (the length of the major axis).

From Figure 146 it follows that when the circle is turned about its diameter A_1A_2 through an angle α, this diameter, being parallel to the H plane, preserves its length in the horizontal projection and becomes the major axis of the ellipse (see Fig. 146, right). As to the diameter B_1B_2, it is obvious that, being turned inclined to the H plane at an angle α, it is projected on this plane shortened: $b_1b_2 = b_1'b_2' \cos\alpha$. This corresponds to the ratio of the axes of the ellipse, i.e. to its compression coefficient.

If two mutually perpendicular diameters are drawn in a circle, then in its projection, which is an ellipse (Fig. 146, right), the projections of such diameters of a circle turn out to be *conjugate* diameters of an ellipse. If in a circle (Fig. 146, left) we draw, for instance, a chord m_1n_1 parallel to the diameter ef, then the diameter cd will bisect this chord and all chords which are parallel to it. Obviously, in an ellipse this property is preserved (see Fig. 146, right): the diameter cd bisects the chord m_1n_1 which is parallel to the diameter ef conjugate to cd. But just *such two diameters of an ellipse either of which bisects the chords which are parallel to the other diameter are conjugate diameters.*

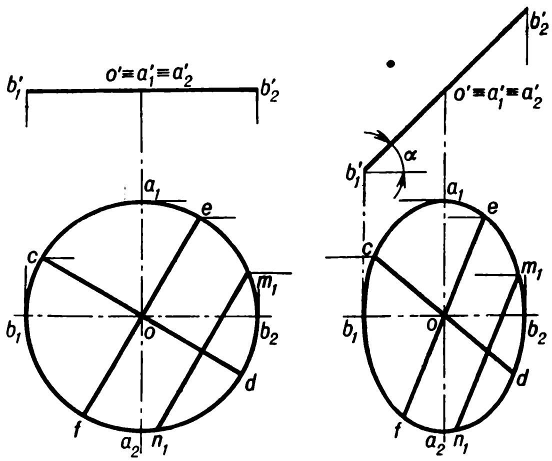

Fig. 146

Conjugate diameters of the ellipse are not perpendicular to each other, its axes, which are also a pair of conjugate diameters, being an exception.

How an ellipse is constructed by its axes is shown in Fig. 147 (left). As is obvious, two concentric circles are drawn, their radii being a (the length of the major axis) and b (the length of the minor axis). If we draw a radius Om_1 and straight lines m_1m_0 and em parallel to the minor and to the major axis, respectively, then the intersection of these lines will yield a point m which belongs to the ellipse. Indeed,

$$\frac{mm_0}{m_1m_0} = \frac{Oe}{Om_1} = \frac{b}{a}.$$

Drawing a number of radii and repeating the above construction, we get a number of points belonging to the ellipse.

Each time a point is constructed, we may obtain three points more which are situated symmetrically to the constructed one about the axes of the ellipse or its centre.

Figure 147 illustrates the construction of the foci of an ellipse: striking from the point B_1 an arc of radius equal to the length of the semimajor axis OA_1, we intersect the major axis at points F_1 and F_2 which are the foci of the ellipse. We then construct an angle F_1KF_2, where K is any point on the ellipse, and draw the bisector in it. Now at point K we draw a tangent line to the ellipse which is perpendicular to the bisector. The line KN perpendicular to the tangent line is a normal to the ellipse at point K.

How are the axes of an ellipse constructed if its conjugate diameters are known?

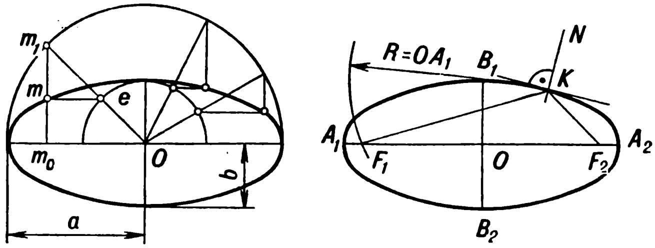

Fig. 147

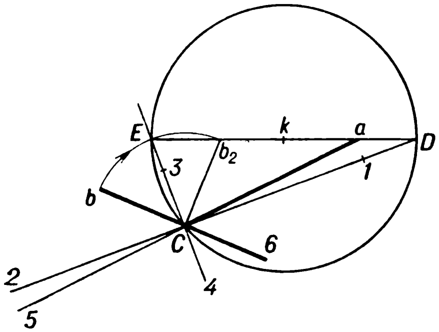

Fig. 148

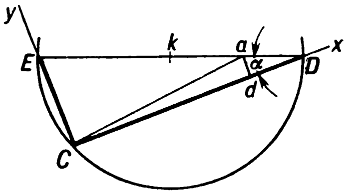

Fig. 149

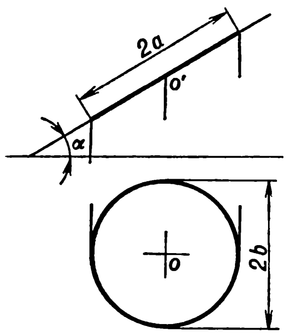

Fig. 150

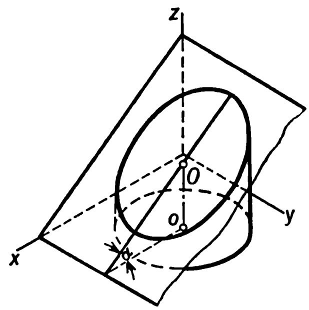

Fig. 151

Suppose the conjugate semidiameters *Ca* and *Cb* are obtained (Fig. 148). Then the axes are constructed in the following succession:

(1) one of the conjugate semidiameters, say *Cb*, is rotated through an angle of 90° towards the other (until the position Cb_2 is occupied);

(2) the line segment ab_2 is drawn and bisected;

(3) a circle of radius *kC* is drawn from point *k*;

(4) the line defined by the segment ab_2 is extended to intersect this circle at points *D* and *E*;

(5) a straight line *DC* is drawn to get the direction of the major axis;

(6) *EC* is drawn to indicate the direction of the minor axis;

(7) $C1 = aE$ is laid off to obtain the major axis;

(8) $C3 = aD$ is laid off to get the minor axis;

(9) the following segments are laid off: $C2 = C1$, $C4 = C3$, $C5 = Ca$, $C6 = Cb$.

The required ellipse can be drawn through the following eight points: *1*, *a*, *3*, *b*, *2*, *5*, *4*, and *6*, or constructed with the aid of the major and minor axes as is shown in Fig. 147.

Thus, with the straight lines *CD* and *CE* drawn, we have the directions of both axes of the ellipse. The point *a* belonging to the ellipse divides the diameter *ED* into two segments one of which (*aE*) is equal to the semimajor axis of this ellipse, the other (*aD*) being equal to the semiminor axis. If (Fig. 149) the lines *CD* and *CE* are taken for the coordinate axes *x* and *y*, respectively, and a perpendicular *ad* is drawn from the point *a* to the line *CD*, then the coordinates of the point *a* may be expressed in the following way

$$x_a = aE \cos \alpha, \quad y_a = aD \sin \alpha.$$

Hence

$$\frac{x_a^2}{(aE)^2} + \frac{y_a^2}{(aD)^2} = \cos^2 \alpha + \sin^2 \alpha = 1.$$

This is the equation of the required ellipse in which *aE* is the semimajor axis, and *aD* is the semiminor axis.

The construction of the horizontal projection of a circle contained in a vertical projecting plane inclined to the *H* plane was shown in Fig. 146. Let now an ellipse with the semiaxes *a* and *b* lie in such a plane. Its projection may sometimes turn out to be a circle with the diameter equal to the minor axis of the ellipse. This will happen when the angle between the plane containing the ellipse and the *H* plane is defined by the relationship $\cos \alpha = \frac{b}{a}$ (Fig. 150). The obtained circle will serve as the projection of a number of ellipses if the angle α and the dimension *a* are varied, and *b* is kept unchanged. Let us imagine a right circular cylinder with a vertical axis (Fig. 151); the inclined sections of this cylinder are ellipses whose minor axis is equal to the diameter of the cylinder.

QUESTIONS TO SECS. 20-21

1. How is a vertical projecting plane drawn through an oblique line represented in the drawing?

2. How are the projections of the centroid constructed in the given drawing of a triangle?

3. What can the projections of a circle represent depending on the position of its plane relative to the projection planes?

4. May we consider the ellipse as a "compressed" circle?

5. What is the compression coefficient of an ellipse?

6. Does the ellipse have: (a) axes of symmetry, (b) a centre of symmetry?

7. What diameters of the ellipse are called: (a) the axes, (b) conjugate diameters?

8. How are the axes of an ellipse constructed given its conjugate diameters?

CHAPTER 4

THE RELATIVE POSITIONS OF TWO PLANES. THE RELATIVE POSITIONS OF A STRAIGHT LINE AND A PLANE

Sec. 22. A Review of the Relative Positions of Two Planes, of a Straight Line and a Plane

Two planes may be mutually parallel or intersect.

Let us consider the case of parallelism of two planes. *If the planes P and Q are parallel* (Fig. 152), *then it is always possible to construct in either of them two intersecting lines so that the lines of one plane are respectively parallel to the two lines of the other.*

This is the main test in determining whether two planes are parallel or not. The traces of both planes may, for instance, serve as such intersecting lines: *if two intersecting traces of one plane are parallel to like traces of the other plane, then the planes are mutually parallel* (Fig. 153, where P_h is parallel to Q_h, and P_v is parallel to Q_v).

Figure 154 demonstrates parallel vertical projecting planes specified by triangles *ABC* and *DEF*. Their parallelism is determined by the parallel vertical projections $a'b'c'$ and $d'f'e'$. If these planes are represented by their traces on *V* and *H*, then, the same as in Fig. 153, both the vertical and the horizontal traces will be respectively parallel. Obviously, if it is known that two parallel lines are vertical projecting, then in some cases we may confine ourselves to giving in the drawing only their vertical traces, as is shown below in Fig. 166 (T_{1v} is parallel to T_{2v}). For horizontal projecting planes (if it is known that they are mutually parallel) it is sufficient in analogous cases to draw their horizontal traces parallel to each other.

Let us consider the case of mutual intersection of two planes. If the planes are specified by their traces, then it is easy to find out that these planes intersect: *if at least one pair of like traces intersect, then the planes intersect.* For instance, in Fig. 155 P_v is parallel to Q_v, but P_h and Q_h intersect: the planes *P* and *Q* intersect.

The foregoing refers to planes specified by *intersecting traces.* But if both planes have on *H* and *V* traces parallel to the *x*-axis, then these planes may either intersect, or be parallel. To judge about the relative positions of such

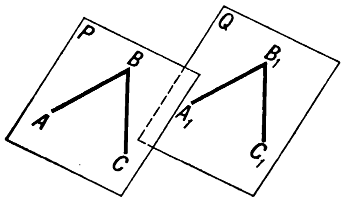

Fig. 152

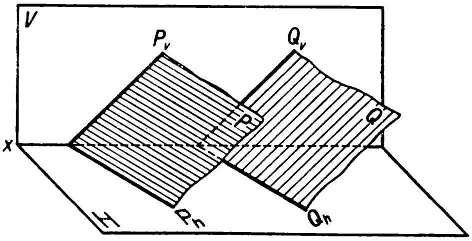

Fig. 153

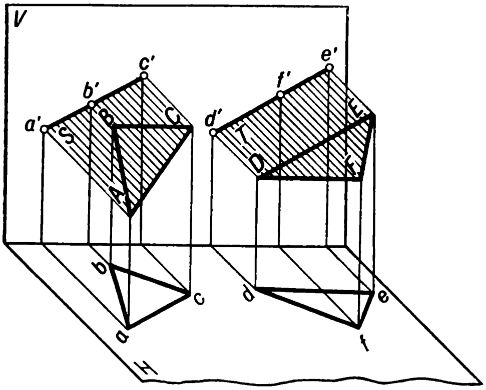

Fig. 154

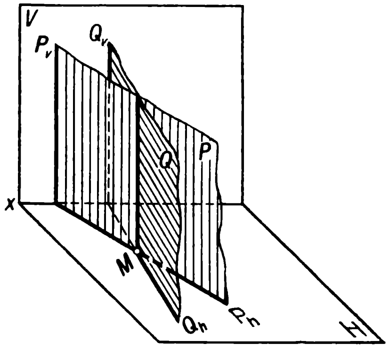

Fig. 155

planes it is advisable to construct their third traces: if the traces of both planes on the third plane of projection are also parallel, then the planes are parallel (Fig. 156: $Q_h || R_h$, $Q_v || R_v$, and $Q_w || R_w$); but if their third traces intersect, then the planes intersect (as in Fig. 157). Obviously, if the traces parallel to the x-axis are arranged, for instance, in such a succession: R_v, Q_v, R_h, Q_h, then the planes cannot be parallel, and therefore there is no need to construct the traces R_w and Q_w.

This is how the problem of the relative positions of two planes is solved when they are *specified by their traces*. Now *if planes are specified not by traces*, but in some other way, and it is required to find out whether they intersect, then, in general, *we have to resort to some auxiliary constructions*. Examples of such constructions will be given below.

Let us now consider the *relative positions of a straight line and a plane in space*. The following cases are possible: (a) the line belongs to the plane, (b) the line intersects the plane, and (c) the line is parallel to the plane.

If it is impossible to determine the relative positions of a line and a plane directly from the drawing, then it is recommended to use some auxiliary constructions which make it possible to reduce the problem of the relative

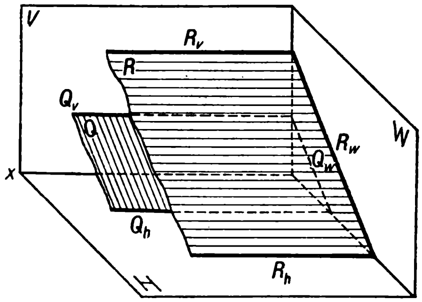

Fig. 156

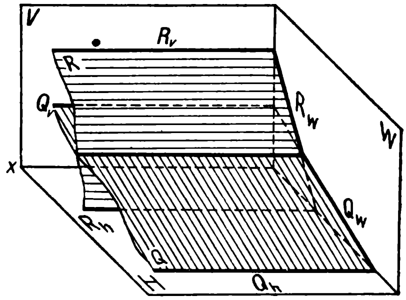

Fig. 157

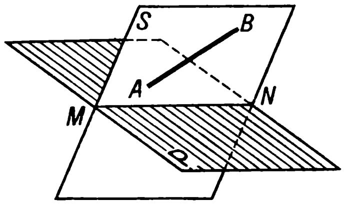

Fig. 158

positions of a line and a plane to the problem of the relative positions of the given line and an auxiliary line. To this effect in Fig. 158 we pass an auxiliary plane *S* through the given line *AB* and establish the relative positions of the two lines *AB* and *MN*, the latter being the line of intersection of the auxiliary plane *S* and the given plane *P*. Three cases are possible here:

(1) The lines *MN* and *AB* coincide; this corresponds to the case of *AB* belonging to the plane *P*.

(2) The line *MN* intersects the line *AB*; this corresponds to the case of *AB* cutting the plane *P*.

(3) The line *MN* is parallel to the line *AB*; this corresponds to the case of *AB* being parallel to the plane *P*.

Hence, *the considered method of determining the relative positions of a straight line and a plane consists in the following:*

(1) *An auxiliary plane is passed through the given line and the line of intersection of this plane and the given plane is constructed;*

(2) *The relative positions of the given line and the line of intersection of the planes are established; the found positions determine the relative positions of the given line and plane.*

This *method of auxiliary planes* is widely used in carrying out constructions when determining the relative positions of various surfaces and of lines and surfaces.

Auxiliary planes are usually chosen so as to simplify all necessary constructions. It may sometimes turn out, for instance, that horizontal or vertical planes, horizontal- and vertical-projecting planes, which are, in general, quite suitable as auxiliary planes, cannot be used at all or their usage will make the appropriate constructions rather complicated even if compared with oblique planes taken as auxiliary planes. When auxiliary planes are to be used in solving this or that problem, they must be adequately chosen so that the constructions involved are as simple as possible and their number is reduced to minimum.

Sec. 23. The Intersection of a Straight Line with a Plane Perpendicular to One or Two Planes of Projection

A plane perpendicular to a projection plane is projected on this plane into a straight line. The corresponding projection of the point in which some straight line cuts such plane must lie on this line (which is the projection of the plane).

In Figure 159 the vertical projection k' of the point of intersection of the line AB with the triangle CDE is determined as the intersection of the projections $a'b'$ and $c'e'$, since the triangle is projected on the V plane into a straight line. Knowing the point k', we determine the position of the projection k. Since a portion of the line AB in the direction from K to B is found under the triangle and therefore is invisible, in the drawing this portion of the line is drawn in a dashed line.

In Figure 160 the vertical trace of the plane T is its vertical projection. The projection k' is determined as the intersection of the projection $a'b'$ and the trace T_v.

Figure 161 gives an example of constructing the projections of the point of intersection of a straight line with a horizontal projecting plane.

For the sake of obviousness the projections of segments of the straight line cutting the plane are represented in different ways: some of them are drawn in continuous lines, the others in dashed lines. This is done with the following considerations in mind:

1. The given plane is conventionally considered to be opaque, that is why all points and lines located with respect to the viewer beyond the plane will be *invisible* though they are situated in the first quadrant. Points and lines are *visible* if they are located on viewer's side of the plane. Let us agree that the viewer is located in the first octant at an infinite distance from the corresponding plane of projection.

2. Visible line segments are drawn with continuous lines, and invisible ones with dashed lines.

3. When a straight line cuts a plane, a portion of this line becomes invisible for the viewer. The point of intersection of a line and a plane separates the line into visible and invisible portions.

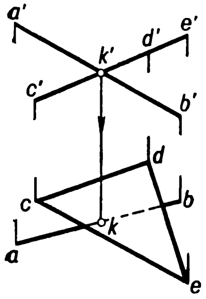

Fig. 159

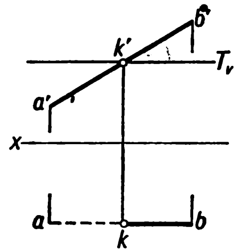

Fig. 160

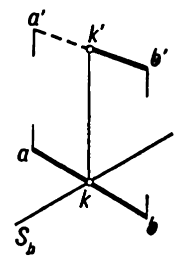

Fig. 161

4. The problem of visibility of a line may always be reduced to the problem of visibility of points. Furthermore, not only a plane can hide a point; a point may hide another point as well (see Fig. 87).

5. *If several points are situated on a common projecting line, then only one of them will be visible:*

(a) with respect to the H plane—this will be the point most distant from H;

(b) with respect to the V plane—the point most distant from V;

(c) with respect to the W plane—the point most distant from W.

6. If we are given a drawing with the axes of projection, then visibility of points lying on a common projecting line is determined by the distances of their corresponding projections from the projection axis:

(a) with respect to the H plane the visible point is one whose vertical projection is farther from the x-axis;

(b) with respect to the V plane the visible point is one whose horizontal projection is farther from the x-axis;

(c) with respect to the W plane the visible point is one whose horizontal projection is most distant from the y-axis.

How must we proceed if a drawing has no axes of projection? Let us consider Fig. 162. Points *1* and *2* of two skew lines are situated on a common projecting line perpendicular to the V plane, and points *3* and *4* on a projecting line perpendicular to the H plane.

The point of intersection of the horizontal projections of the given lines represents the merged projections of two points of which point *4* belongs to the line AB, and point *3* to the line CD. Since *3'3* is longer than *4'4*, point *3* belonging to the line CD is visible with reference to the H plane, while point *4* is hidden by point *3*.

The point of intersection of the vertical projections of the lines AB and CD also represents merged projections of two points *1* and *2*, of which point *1* belongs to the line AB, and point *2* to the line CD. Since *11'* is longer than *22'*, we conclude that point *1* is visible with reference to the V plane, thus hiding point *2*.

This is a general method, and it is applicable to drawings supplied with the projection axes as well.

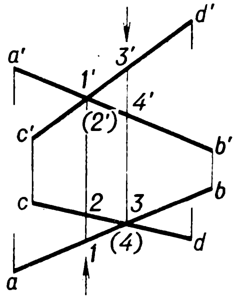

Fig. 162

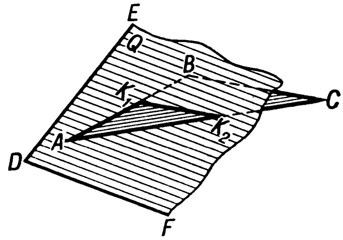

Fig. 163

Sec. 24. Constructing the Line of Intersection of Two Planes

A straight line obtained as mutual intersection of two planes is completely defined by two points either of which belongs to both planes.

For instance, the straight line K_1K_2 (Fig. 163) of intersection of the plane specified by the triangle ABC and the plane Q specified by two lines DE and DF passes through the points K_1 and K_2; but just at these points the lines AB and AC of the first plane intersect the plane Q, i.e. the points K_1 and K_2 belong to both planes.

Consequently, *in the general case, to construct the line of intersection of two planes we have to find two arbitrary points either of which belongs to both planes; these points will define the line of intersection of the planes.*

To find the two required points we have usually to carry out special constructions. But if at least one of the intersecting planes is perpendicular to a plane of projection, then the construction of the projections of the line of intersection is simplified. Let us begin with such a case.

Figure 164 shows the intersection of two planes one of which (specified by the triangle DEF) is perpendicular to the V plane. Since the triangle DEF is projected on the V plane into a straight line ($d'f'$), the vertical projection of the line of intersection of the two given triangles represents a line segment $k_1'k_2'$ on the projection $d'f'$. The further construction is clear from the drawing.

Another example is given in Fig. 165: a horizontal projecting plane S intersects the plane of the triangle ABC. The horizontal projection of the line of intersection of these planes (the line segment mn) is determined on the trace S_h.

Now we are going to consider the *general case of constructing the line of intersection of two planes.*

Let one of the planes (P) be specified by two intersecting lines, and the other (Q) by two parallel lines. The appropriate construction is shown in Fig. 166. The line K_1K_2 is obtained as a result of mutual intersection of the planes P and Q, or in brief notation: $P \times Q = K_1K_2$.

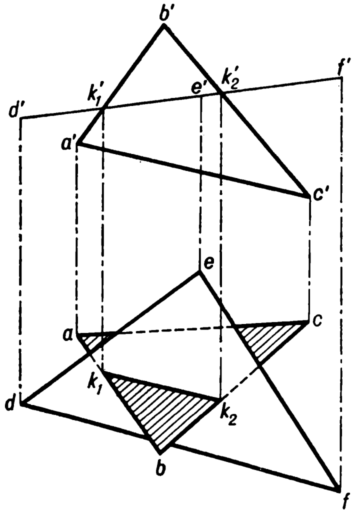

Fig. 164

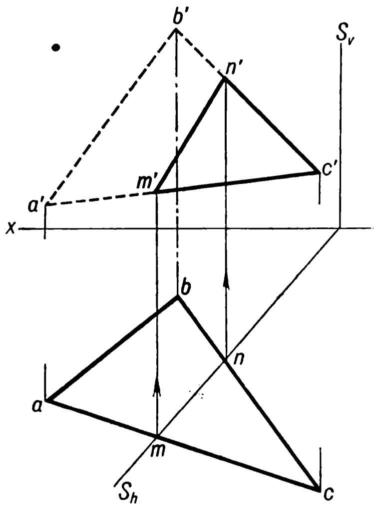

Fig. 165

To determine the positions of the points K_1 and K_2 we take two auxiliary vertical projecting planes (T_1 and T_2) which intersect either of the planes P and Q. The plane T_1, intersecting the planes P and Q, yields two straight lines whose projections are *1'2'*, *1-2* and *3'4'*, *3-4*. The intersection of these lines contained in the plane T_1 determines the first point (K_1) defining the line of intersection of the planes P and Q.

Introducing then the plane T_2 we obtain the lines of its intersection with the planes P and Q. In the drawing their projections are: *5'6'*, *5-6* and *7'8'*, *7-8*. The intersection of these lines contained in the plane T_2 determines the second point (K_2) common to P and Q.

Having now the projections k_1 and k_2, we find the projections k_1' and k_2' on the traces T_{1v} and T_{2v}. This determines the projections k_1k_2 and $k_1'k_2'$ of the required line of intersection of the planes P and Q (the projections thus determined are drawn with a dot-and-dash line).

When carrying out the construction the following should be borne in mind: since the auxiliary secant planes T_1 and T_2 are mutually parallel, then, on constructing the projections *1-2* and *3-4*, we take only one point for either of the projections *5-6* and *7-8*, say *5* and *8*, since *5-6* is parallel to *1-2* and *7-8* is parallel to *3-4*.

In the above construction we took two vertical projecting planes as auxiliary planes. Of course, we might take some other planes as well, for instance, two horizontal planes, or one horizontal and one vertical planes, and so on. But all this does not change the nature of construction. We may come across the following case. Suppose two horizontal planes were taken as auxiliary planes, and the horizontal lines obtained as their intersection with

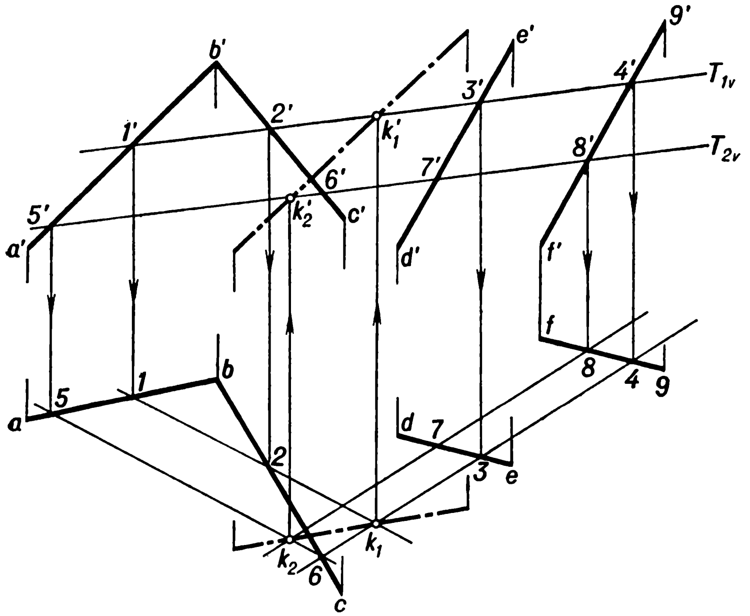

Fig. 166

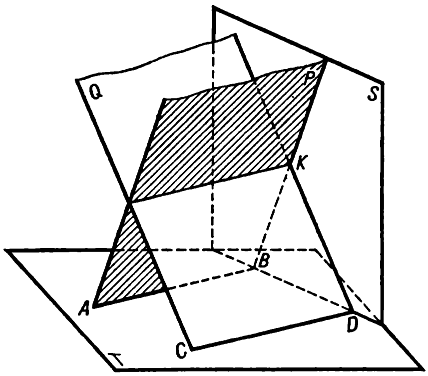

Fig. 167

the planes P and Q turned out to be mutually parallel. Fig. 167 shows that P and Q intersect, though their horizontal lines are parallel. Hence, having obtained parallel horizontal projections of the horizontal lines AB and CD and knowing that the planes are not necessarily parallel (they may, for instance, intersect along a common horizontal line), we have to test the planes P and Q, using for this purpose, say, a horizontal projecting plane (see Fig.

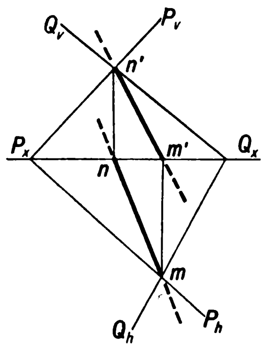

Fig. 168

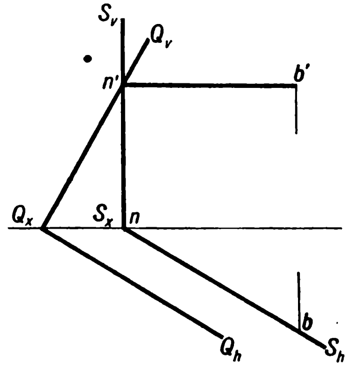

Fig. 169

167): if the lines along which the auxiliary plane S intersects P and Q would appear to be also parallel, then the planes P and Q do not intersect—they are parallel to each other. In Figure 167 these lines intersect at point K, through which there just passes the line of intersection of the planes P and Q parallel to the lines BA and CD.

If planes are specified by their traces on the projection planes, then it is natural to find the points determining the lines of intersection of the planes at the points of intersection of like traces of the planes (Fig. 168): *the line passing through these points is common to both planes, i.e. is the line of their intersection.*

The above-considered method on constructing the line of intersection of two planes (see Fig. 166) may, of course, be used in the case when the planes are specified by their traces. Here the role of auxiliary secant planes is played by the projection planes themselves:

$$P \times H = P_h; \quad Q \times H = Q_h; \quad P_h \times Q_h = M;$$
$$P \times V = P_v; \quad Q \times V = Q_v; \quad P_v \times Q_v = N.$$

The points of intersection of like traces of planes are the traces of the line of intersection of these planes. Therefore, to construct the projections of the line of intersection of the planes P and Q (Fig. 168) proceed as follows: (1) find m as the point of intersection of the traces P_h and Q_h and n' as the point of intersection of the traces P_v and Q_v; using m and n', determine the projections m' and n; (2) draw straight lines $m'n'$ and mn.

Figures 169-171 give a few examples when the direction of the line of intersection is known. In this case it is sufficient to have only one point of intersection of the traces and then to draw a line through this point, proceeding from the positions of the planes and their traces.

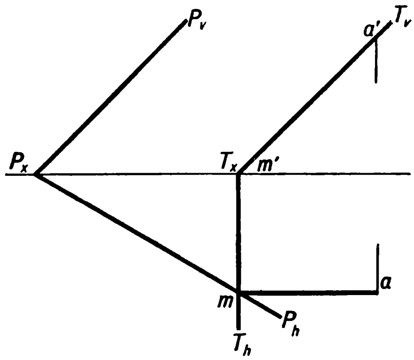

Fig. 170

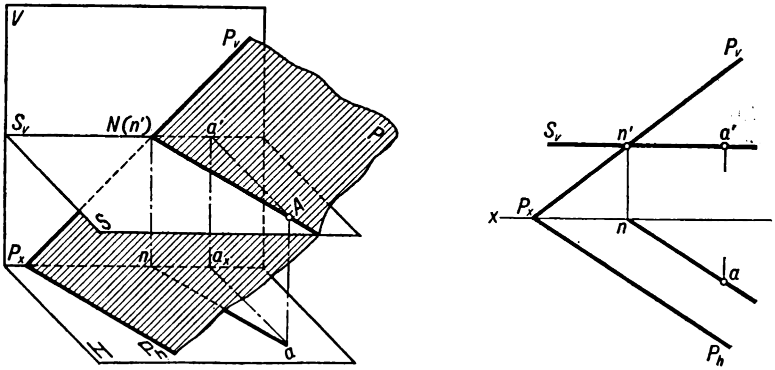

Fig. 171

QUESTIONS TO SECS. 22-24

1. What are the relative positions of two planes?

2. What is the test for parallelism of two planes?

3. What are the relative positions of the vertical traces of two parallel vertical projecting planes?

4. What are the relative positions of the horizontal traces of two parallel horizontal projecting planes?

5. What are the relative positions of like traces of two parallel planes?

6. Does the intersection of at least one pair of like traces of two planes serve as a test for intersection of these planes?

7. How may we determine the relative positions of a straight line and a plane?

8. How do we construct the line of intersection of a straight line and a plane perpendicular to one or two planes of projection?

9. What point of those situated on a common perpendicular to (a) the *H* plane, (b) the *V* plane is considered to be visible on the respective planes?

10. How do we construct the line of intersection of two planes at least one of which is perpendicular either to *H* or to *V*?

11. What does the general method of constructing the line of intersection of two planes consist in?

Sec. 25. The Intersection of a Straight Line with an Oblique Plane

To construct the point of intersection of a straight line and an oblique plane proceed as follows (Fig. 158):

(1) *draw an auxiliary plane (S) through the given line (AB);*

(2) *construct the line (MN) of intersection of the given (P) and auxiliary (S) planes;*

(3) *determine the position of the point (K) of intersection of the given (AB) and constructed (MN) lines.*

Fig. 172 shows the construction of the point of intersection of a straight line *FK* with an oblique plane specified by two intersecting lines *AB* and *CD*.

An auxiliary vertical projecting plane *S* is passed through the line *FK*. The choice of a vertical projecting plane is explained by the fact that in this case it is convenient to construct the points of intersection of its vertical trace with the projections $a'b'$ and $c'd'$. Using the points m' and n', we find the horizontal projections *m* and *n* and thus determine the line *MN* along which the auxiliary plane *S* intersects the given plane *P*. We then find the point *k* at which the horizontal projection of the line directly, or when extended, intersects the projection *mn*. Finally, we determine the vertical projection of the point of intersection, i.e. the point k'.

Figure 173 illustrates the construction of the point of intersection of a line *MN* with a plane specified by a triangle *ABC*. The succession of construction follows the pattern of Fig. 172, the only difference being that the auxiliary plane (this time a horizontal projecting one) here is indicated only by one trace T_h passing through the projection *mn*. The plane *T* cuts the triangle *ABC* along the line *DE*. But we can do without T_h as well: imagining the auxiliary horizontal projecting plane passing through *MN*, we represent by *ed* and $e'd'$ the line segment *ED* along which the horizontal projecting plane passed through *MN* intersects the triangle.

Considering that a straight line and an opaque triangle are given in space, let us determine the visible and invisible portions of the line *MN* with respect to the planes *H* and *V*.

There are two coincident horizont al projections of two points at point *e* on the *H* plane, one of which (the vertical projection e_1') belongs to the

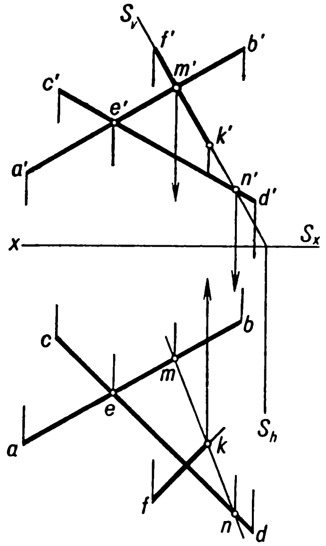

Fig. 172

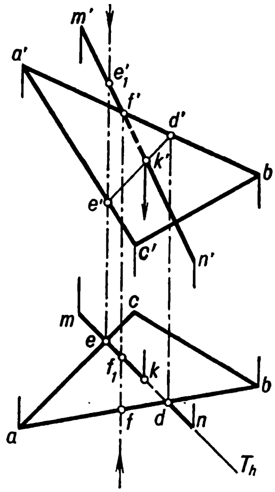

Fig. 173

line MN, and the other (the vertical projection e') to the side AC of the triangle.

From the location of the vertical projections e_1' and e' it follows that the portion KM of the line is above the triangle and, hence, on the horizontal projection the line segment mk is entirely visible, whereas the line segment kd is invisible.

On the vertical projection the point f' represents the coincident projections of two points, one of which belongs to the line MN, the other belonging to the side AB of the triangle. Judging by the location of the horizontal projections f and f_1, we conclude that the portion MK of MN is behind the triangle and, hence, on the vertical projection the line segment $f'k'$ is invisible, and the line segment $k'n'$ is visible.

Figures 174-176 give examples of constructing the point of intersection of a straight line and an oblique plane represented by its traces. In the first example through the line AB we draw a horizontal projecting plane S, and in the second (Fig. 175), a horizontal plane. It turns out to be possible to proceed this way, since in the second case AB is a horizontal line.

The straight line represented in Fig. 176 is perpendicular to the H plane, therefore the horizontal projections of all points of this line merge into one point. Hence, the position of the projection k of the required point of intersection of the line AB with the plane P is known. The position of the projection k' is determined with the aid of a horizontal line.

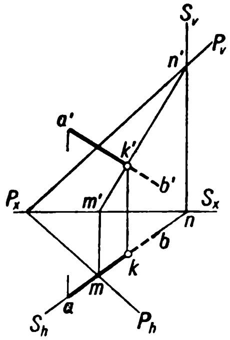

Fig. 174

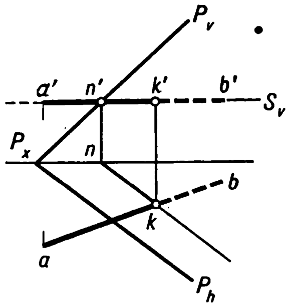

Fig. 175

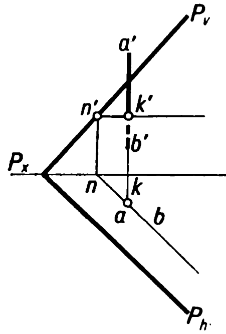

Fig. 176

Sec. 26. Constructing the Line of Intersection of Two Planes Using the Points of Intersection of Straight Lines with a Plane

In Section 24 we discussed the general method for constructing the line of intersection of two planes, namely, the use of auxiliary secant planes (see Fig. 166). Let us now consider another method of construction as applied to oblique planes. *This method consists in that we find the points of intersection of two straight lines belonging to one of the planes with the other plane.* Consequently, the student should know how to construct the point of intersection of a straight line and an oblique plane, which is set forth in Sec. 25.

Figure 177 shows the triangle *ABC* intersected by a plane specified by two parallel lines (*DE* and *FG*). The construction is reduced to constructing the points K_1 and K_2 at which the lines *DE* and *FG* cut the plane containing the triangle and to drawing a line segment through these points. Imagining vertical projecting planes drawn through *DE* and *FG*, we find the parallel lines along which these planes intersect the triangle. One of them is represented by the projections *1-2* and *1′2′*, the other only by one point *3′*, *3* through the horizontal projection of which a straight line is drawn parallel to the projection *1-2*.

On determining the projections k_1 and k_2 we find the projections k_1' and k_2' and the projection of the line segment k_1k_2.

Of course, to the case under consideration the general method is applicable as well (see Fig. 166), but we would then have to draw more lines, than it has been done in Fig. 177.

Figure 178 demonstrates the construction of the line of intersection of two triangles *ABC* and *DEF* with visible and invisible portions of these triangles indicated.

The line K_1K_2 is constructed using the points of intersection of the sides *AC* and *BC* of the triangle *ABC* with the plane containing the triangle *DEF*. An auxiliary vertical projecting plane drawn through *AC* (it is not shown in the drawing) intersects the triangle *DEF* along a straight line represented

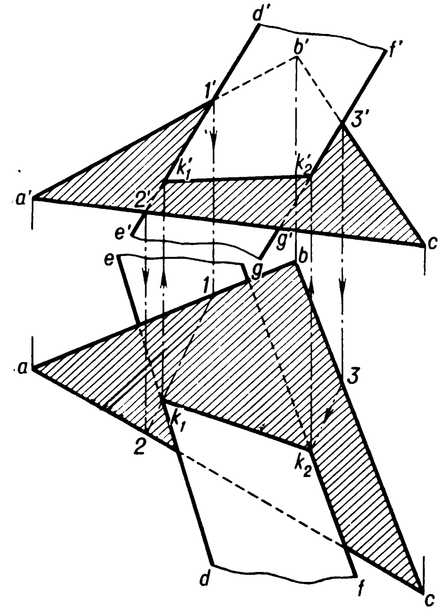

Fig. 177

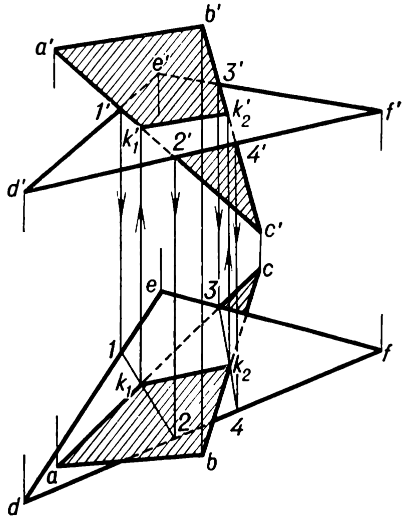

Fig. 178

by the projections *1'2'* and *1-2*; the intersection of the projections *ac* and *1-2* yields the horizontal projection of the point K_1 of intersection of the line *AC* and the triangle *DEF*; then the vertical projection k_1' is constructed. The point K_2 is found in the same way.

The examples given in Figs. 177 and 178 raised the problem of separating plane figures into portions, visible and invisible for the viewer, since planes are considered conventionally to be opaque. The visible portions of the triangles *ABC* are hatched in the mentioned drawings (see also Fig. 173).

Figure 179 gives one more example of construction of the line of intersection of two triangles. In this case we may assume with equal grounds that the triangle *ABC* enters the slit made in the triangle *DEF*, or that the triangle *DEF* goes into the slit out in the triangle *ABC*. We have to agree: to what triangle this slit (along the line K_1K_2) belongs. Quite another thing with the triangles shown in Fig. 178: a slit is obviously out in the triangle *DEF* and the triangle *ABC* passes through this slit, thus cutting the triangle *DEF*.

The construction itself in Fig. 179 is reduced to finding the points K_1 and K_2 with the aid of vertical projecting planes P_1 and P_2, respectively.

We would like to underline once again that the use of dashed lines instead of continuous ones, as for example, in Figs. 159, 161, 164, 165 and 173 to 179, is prompted by the desire to make the drawings more descriptive. If we proceed from the notion of a projection as a geometric image, then the question of 'transparency' or 'opacity' would no longer arise, of 'visibility'

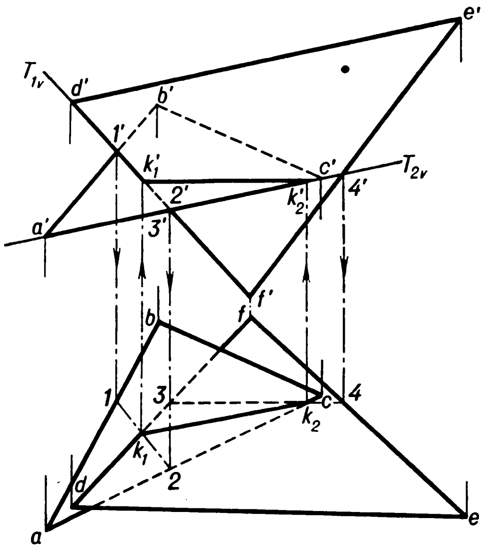

Fig. 179

or 'invisibility': everything should be drawn with continuous lines. But to make drawings more obvious some conventions are introduced, the use of dashed lines being one of them.

QUESTIONS TO SECS. 25-26

1. What does the method for constructing the point of intersection of a straight line and a plane consist in?

2. What is the procedure of constructing this point (see Question 1)?

3. How is 'visibility' determined in the case of a line cutting a plane?

4. How is it possible to construct the line of intersection of two planes without using the general method described in Sec. 24?

5. How is 'visibility' determined in the case of mutual intersection of two planes?

6. In what respect do the cases considered in Figs. 178 and 179 differ from each other?

Sec. 27. Constructing a Straight Line and a Plane Parallel to Each Other

The construction of a *straight line parallel to a given plane* is based on the following statement known from geometry: *a straight line is parallel to a plane if this line is parallel to any straight line contained in the plane.*

Through a given point in space we can draw an infinite number of straight

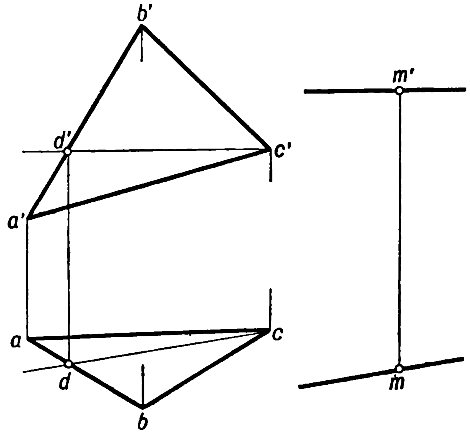

Fig. 180

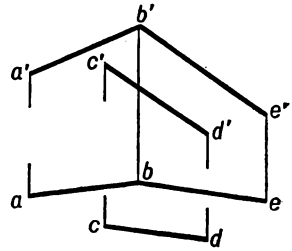

Fig. 181

lines parallel to a given plane. To obtain a unique solution an additional condition is required. For instance, it is required to draw through the point *M* (Fig. 180) a straight line parallel to both the plane specified by the triangle *ABC* and the *H* plane of projection (the latter being the additional condition).

Obviously, the required line must be parallel to the line of intersection of both planes, i.e. must be parallel to the horizontal trace of the plane specified by the triangle *ABC*. In Figure 180 a horizontal line *DC* is used and then, parallel to this line, a straight line is drawn through the point *M*.

Let us solve the inverse problem: *through a given point draw a plane parallel to a given straight line.* The planes passing through a point *A* parallel to a line *BC* form a pencil of planes whose axis is a straight line passing through the point *A* and parallel to the line *BC*. And again, to get the unique solution a certain additional condition is required.

For instance, it is required to draw a plane parallel to the line *CD* not through a point, but through a straight line *AB* (Fig. 181). As is clear, *AB* and *CD* are skew lines. If it is required through one of the two skew lines to pass a plane parallel to the other line, then the problem has a unique solution. Through the point *B* a straight line is drawn parallel to the line *CD*; the lines *AB* and *BE* determine then the plane parallel to the line *CD*.

How can we find out whether a given line is parallel to a given plane?

We may try to draw in this plane a straight line parallel to the given line. If we fail to construct such a line in the plane, then the given line and plane are not parallel to each other.

We can also try to find the point of intersection of the given line with the given plane. If we fail to find such a point, then the given line and plane are mutually parallel.

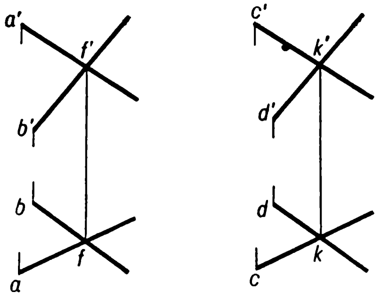

Fig. 182

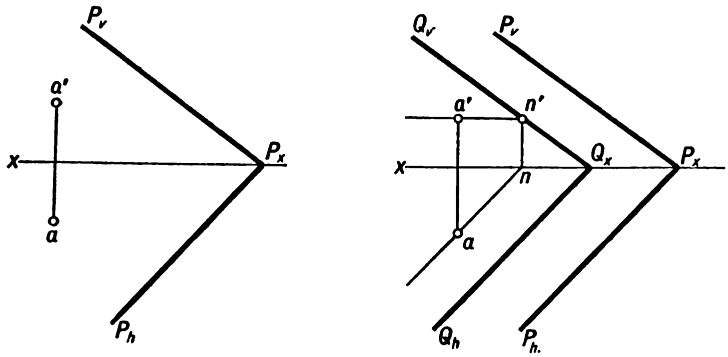

Fig. 183

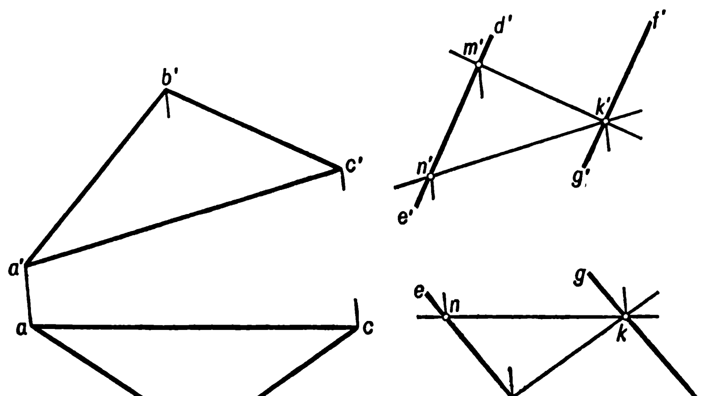

Fig. 184

Sec. 28. Constructing Mutually Parallel Planes

Let there be given a point K through which it is required to pass a plane parallel to a plane specified by two intersecting lines AF and BF (Fig. 182).

Obviously, if through the point K straight lines CK and DK are drawn respectively parallel to the lines AF and BF, then the plane defined by the lines CK and DK will turn out to be parallel to the given plane.

Another example of construction is given in Fig. 183 (right). Through the point A a plane Q is drawn parallel to the plane P. First we draw through the point A a straight line *a fortiori* parallel to the plane P. This is a horizontal line whose projections are $a'n'$ and an, the latter being parallel to P_h. Since point N is the vertical trace of the horizontal line AN, the trace Q_v parallel to P_v will pass through this point and the trace Q_h parallel to P_h through Q_x. Hence, the planes Q and P are mutually parallel, since their like intersecting traces are mutually parallel.

Figure 184 represents two parallel planes, one of them is specified by a triangle ABC, the other by parallel lines DE and FG. What verifies the parallelism of these planes? The fact that in the plane specified by the lines DE and FG we were able to draw two intersecting lines KM and KN respectively parallel to the lines AC and BC contained in the other plane.

Of course, we might try to find the point of intersection of, say, the line DE and the plane of the triangle ABC. The failure would confirm parallelism of the planes.

QUESTIONS TO SECS. 27-28

1. What is the construction of a straight line to be parallel to a certain plane based on?
2. How is a plane drawn through a straight line parallel to a given line?
3. What defines the parallelism of two planes?
4. How is a plane parallel to a given plane passed through a given point?
5. How is parallelism of given planes checked in the drawing?

Sec. 29. Constructing a Straight Line and a Plane Perpendicular to Each Other

Of all possible positions of a straight line intersecting a plane we are going to point out the case when a line is perpendicular to a plane, and consider the properties of such a line.

Figure 185 shows a plane defined by two intersecting lines AN and AM, AN being a horizontal line and AM a vertical line of the given plane. The line AB represented on the same drawing is perpendicular both to AN and AM and, hence, is perpendicular to the plane defined by them.

A perpendicular to a plane is perpendicular to any line drawn in this plane. But for the projection of a perpendicular to an oblique plane to be perpendicular to the like projection of a straight line contained in this

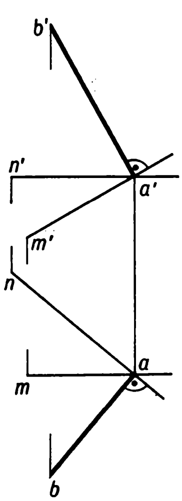

Fig. 185

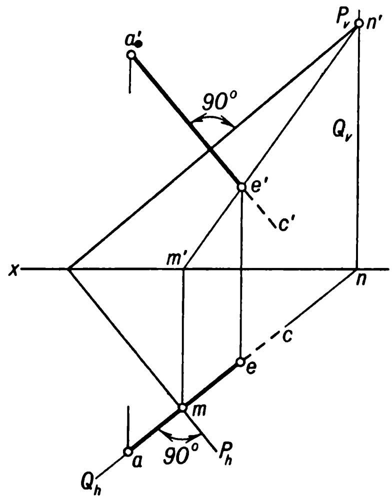

Fig. 186

plane, it is necessary that the straight line is a horizontal line, or a vertical line, or a profile line of the plane. Therefore, when it is desired to construct a perpendicular to a plane, two such lines are taken in the general case (for instance, a horizontal and vertical lines, as is shown in Fig. 185).

Thus, a perpendicular to a plane possesses the following property: *its horizontal projection is perpendicular to the horizontal projection of a horizontal line, its vertical projection is perpendicular to the vertical projection of a vertical line, and its profile projection is perpendicular to the profile projection of a profile line of the plane.*

Obviously, when a plane is represented by its traces (see Fig. 186), we draw the following conclusion: *if a line is perpendicular to a plane, then the horizontal projection of this line is perpendicular to the horizontal trace of the plane, and the vertical projection is perpendicular to the vertical trace of the plane.*

Thus, if in the system *V*, *H* the horizontal projection of a straight line is perpendicular to the horizontal trace of a plane and the vertical projection of the line is perpendicular to its vertical trace, then *in the case of oblique planes* (as in Fig. 186), *as also of horizontal- and vertical-projecting planes, the line is perpendicular to the plane.* But in the case of a profile projecting plane it may happen that a line is not perpendicular to this plane, though the projections of the line are respectively perpendicular to the horizontal and vertical traces of the plane. Therefore, in the case of a profile projecting plane it is necessary to consider also the relative positions of the profile projection of the line and the profile trace of the given plane and only after such an examination to determine whether the given line and plane are mutually perpendicular.

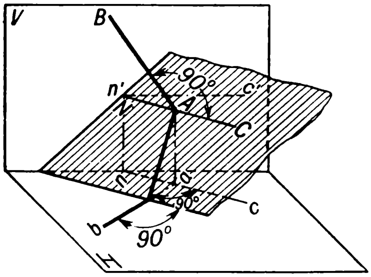

Fig. 187

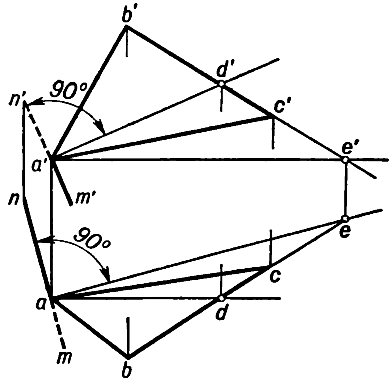

Fig. 188

Obviously, in Fig. 187 the horizontal projection of a perpendicular to a plane coincides with the horizontal projection of a slope line drawn in the plane through the foot of the perpendicular.

In Figure 186 from point A a perpendicular is drawn to plane P ($a'c'$ is perpendicular to P_v, and ac to P_h), and the point E is constructed at which the perpendicular AC cuts the plane P. The construction is accomplished with the aid of horizontal projecting plane Q passed through the perpendicular AE.

Figure 188 illustrates how a perpendicular is dropped to the plane defined by the triangle ABC. The perpendicular is drawn through the point A.

Since the vertical projection of a perpendicular to a plane must be perpendicular to the vertical projection of a vertical line contained in the plane, and its horizontal projection is perpendicular to the horizontal projection of a horizontal line, drawn in the plane through the point A are a vertical line ($a'd'$ and ad) and a horizontal line ($a'e'$ and ae). It goes without saying, that these lines should not necessarily be drawn through the point A.

Then the projections of the perpendicular are drawn: $m'n'$ perpendicular to $a'd'$ and mn perpendicular to ae. Why are the portions $a'n'$ and am of the projections in Fig. 188 drawn with dashed lines? Because here we consider the plane specified by the triangle ABC, but not only this triangle: the perpendicular is located partially before the plane, and partially behind it.

Figures 189 and 190 show the construction of a plane passing through the point A perpendicular to the line BC. In Figure 189 the plane is represented by its traces. We begin the construction with drawing through point A the horizontal line of the required plane: since the horizontal trace of the plane must be perpendicular to bc, then the horizontal projection of the horizontal line must also be perpendicular to bc. Therefore an is perpendicular to bc. The projection $a'n'$ is parallel to the x-axis, since it is a characteristic feature of a horizontal line. We then draw through the point n

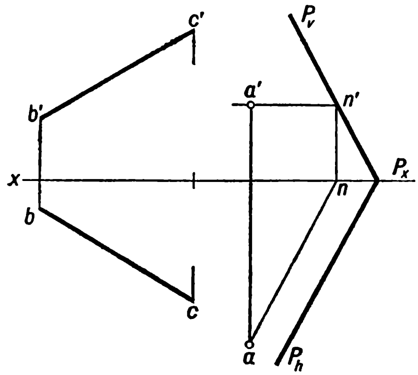

Fig. 189

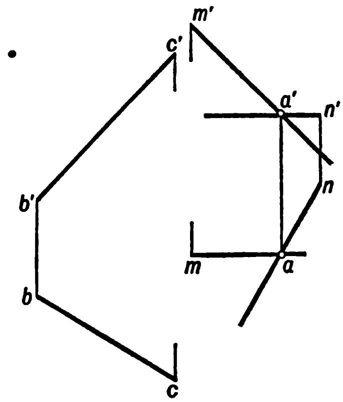

Fig. 190

(which is the vertical projection of the vertical trace of the horizontal line *AN*) the trace P_v perpendicular to *b'c'*, obtain the point P_x, and draw the trace P_h parallel to *an* (P_h is perpendicular to *bc*).

In Figure 190 the plane is defined by its vertical line *AM* and a horizontal line *AN*. These lines are perpendicular to *BC* (*a'm'* is perpendicular to *b'c'*, and *an* to *bc*); the plane defined by them is perpendicular to *BC*.

Since a perpendicular to a plane is perpendicular to any straight line drawn in this plane, then, on learning how to pass a plane perpendicular to a straight line, we can take advantage of this method for drawing a perpendicular from a point *A* to an oblique line *BC*. Obviously, we may plan the following procedure for constructing the required line:

(1) to pass through the point *A* a plane (designated by *Q*) perpendicular to *BC*;

(2) to determine the point *K* of intersection of the line *BC* with the plane *Q*;

(3) to join the points *A* and *K* with a line segment.

The lines *AK* and *BC* are mutually perpendicular.

An example of appropriate construction is given in Fig. 191. Through the point *A* a plane (*Q*) is drawn perpendicular to *BC*. This is done with the aid of a vertical line, whose vertical projection *a'f'* is drawn perpendicular to the vertical projection *b'c'*, and a horizontal line whose horizontal projection is perpendicular to *bc*.

Then the point *K* is found at which the line *BC* pierces the plane *Q*. To this end a horizontal projecting plane *S* (in the drawing it is specified only by the horizontal trace S_h) is drawn through the line *BC*. The plane *S* intersects the plane *Q* along a straight line whose projections are *1'2'* and *1-2*. The intersection of this line with the line *BC* yields the point *K*. The line *AK* is the required perpendicular to *BC*. Indeed, the line *AK* intersects the line *BC* and is contained in the plane *Q* perpendicular to the line *BC*; consequently, *AK* is perpendicular to *BC*.

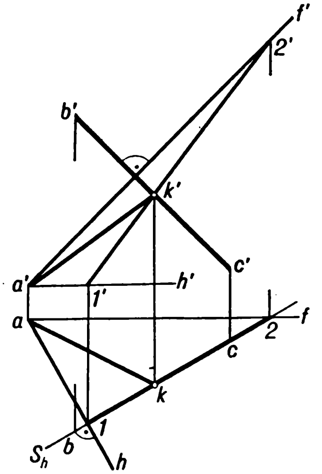

Fig. 191

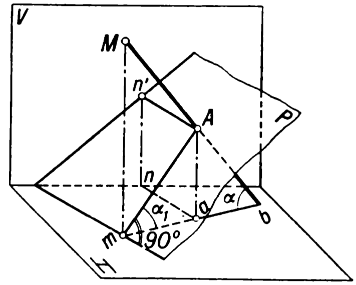

Fig. 192

In Section 15 we showed (Fig. 92) how to draw a perpendicular from a point to a line. It was accomplished by introducing an additional plane into the system V, H, thus forming the system S, H in which the plane S is passed parallel to the given line. We recommend to compare the constructions given in Figs. 92 and 191.

Figure 192 shows an oblique plane P passing through the point A, and a perpendicular AM to this plane extended to intersect the H plane at point b.

The angle α_1 between the planes P and H and the angle α between the line AM and the plane H are the acute angles of a right-angled triangle bAm and, hence, $\alpha_1+\alpha = 90°$. Analogously, if the plane P forms an angle β_1 with the V plane, and the line AM perpendicular to P forms an angle β with the V plane, then $\beta_1+\beta = 90°$. From this first of all it follows that an oblique plane which must form an angle α_1 with the H plane, and an angle β_1 with the V plane may be constructed only if $180° > \alpha_1+\beta_1 > 90°$.

Indeed, adding $\alpha_1+\alpha = 90°$ and $\beta_1+\beta = 90°$ termwise, we get: $\alpha_1+\beta_1+$ $+\alpha+\beta = 180°$, i.e. $\alpha_1+\beta_1 < 180°$, and since $\alpha+\beta < 90°$ (see Sec. 13), we have $\alpha_1+\beta_1 > 90°$. If we take $\alpha_1+\beta_1 = 90°$, then a profile projecting plane is obtained, and if $\alpha_1+\beta_1 = 180°$ is taken, then we get a profile plane, i.e. in both cases we obtain not an oblique plane but planes of particular positions.

Sec. 30. Constructing Mutually Perpendicular Planes

A plane Q perpendicular to a plane P can be constructed in two ways: (1) it is passed through a line perpendicular to the plane P; (2) it is drawn perpendicular to a line contained in the plane P or in a plane parallel to this plane. To get a unique solution additional conditions are required.

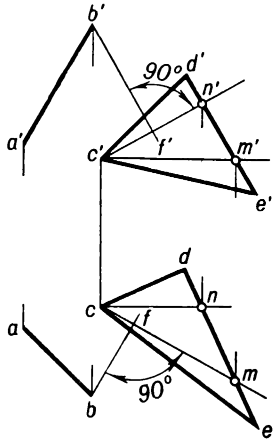

Fig. 193

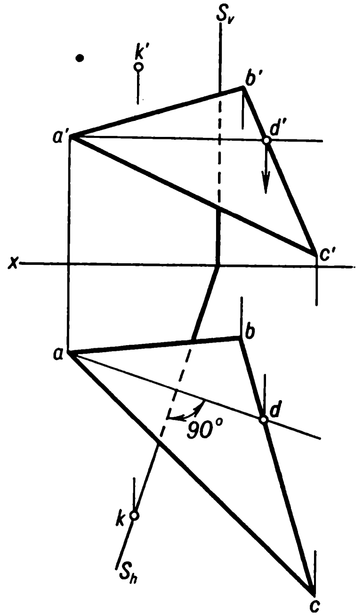

Fig. 194

Figure 193 shows the construction of a plane perpendicular to the plane specified by the triangle *CDE*. An additional condition here is that the required plane must pass through the line *AB*. Consequently, the required plane is defined by the line *AB* and a perpendicular to the plane containing the triangle. To draw this perpendicular to the plane *CDE* we take in it a vertical line *CN* and a horizontal line *CM*: if $b'f'$ is perpendicular to $c'n'$ and bf to cm, then *BF* is perpendicular to the plane *CDE*.

The plane defined by the intersecting lines *AB* and *BF* is perpendicular to the plane *CDE*, since it passes through a perpendicular to this plane. In Figure 194 a horizontal projecting plane *S* passes through the point *K* perpendicular to the plane specified by the triangle *ABC*. Here the additional condition is perpendicularity of the required plane to two planes at once: to the plane *ABC* and to the *H* plane. Therefore the answer is a horizontal projecting plane. And since it is passed perpendicular to the horizontal line *AD*, i.e. to a line belonging to the plane *ABC*, the plane *S* is perpendicular to the plane *ABC*.

May perpendicularity of like traces of planes serve as a test for perpendicularity of the planes themselves?

To obvious cases, when this is so, belongs mutual perpendicularity of two horizontal projecting planes whose horizontal traces are mutually perpendicular.

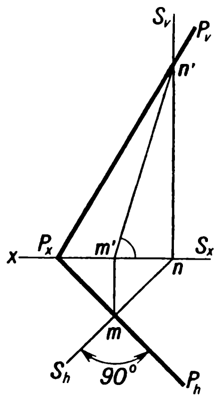

Fig. 195

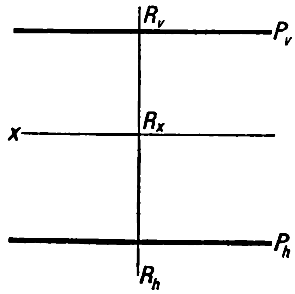

Fig. 196

The same can be said about mutual perpendicularity of the vertical traces of the vertical projecting planes. These traces are mutually perpendicular.

Let us consider (Fig. 195) the horizontal projecting plane S which is perpendicular to an oblique plane P. If the plane S is perpendicular to the planes H and P, then S is perpendicular to P_h as to the line of intersection of the planes P and H. Hence, P_h is perpendicular to S and, consequently, P_h is perpendicular to S_h as to one of the lines contained in the plane S.

Thus, perpendicularity of the horizontal traces of an oblique plane and a horizontal projecting plane corresponds to mutual perpendicularity of these planes.

Obviously, perpendicularity of the vertical traces of a vertical projecting and an oblique planes also corresponds to mutual perpendicularity of these planes.

But *if like traces of two oblique planes are mutually perpendicular, then the planes themselves are not perpendicular to each other*, since none of the conditions set forth at the beginning of this section is met here.

We conclude this section by considering Fig. 196. Here we come across a situation when both pairs of like traces, as also the planes themselves, are mutually perpendicular: both planes occupy particular positions, namely, R is a profile projecting plane.

Sec. 31. Constructing the Projections of an Angle Between a Straight Line and a Plane, and Between Two Planes

If a straight line is not perpendicular to a plane, then *the angle between the line and the plane is defined as an angle between this line and its projection on the given plane.*

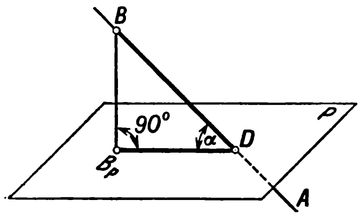

Fig. 197

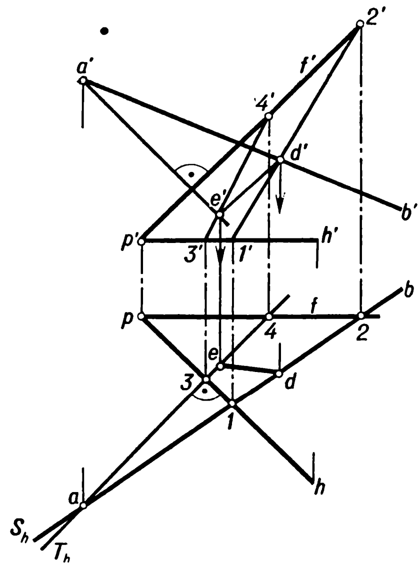

Fig. 198

For angles between a straight line and the projection planes see Sec. 13.

Figure 197 represents a straight line AB intersecting the plane P at point D; the angle α is formed by the segment BD of the given line and the projection B_pD of this segment on the plane P.

The construction of the projections of the angle between a line AB and a plane P is carried out in Fig. 198. The plane P is specified by its horizontal (the projections $p'h'$ and ph) and vertical (the projections $p'f'$ and pf) planes.

The construction is accomplished in the following succession:

(a) the point D of intersection of the line AB with the plane P is found, for which purpose a horizontal projecting plane S is passed through AB;

(b) a perpendicular is drawn from the point A to the plane P;

(c) the point E of intersection of this perpendicular with the plane P is found;

(d) straight lines are drawn through the points d' and e', d and e, thus determining the *projections of the line AB on the plane P.*

The angle $a'd'e'$ represents the vertical projection of the angle between AB and the plane P, and the angle ade the horizontal projection of this angle.

The construction of projections of an angle between a line and a plane is considerably simplified if the plane is not an oblique one, since in such cases the point of intersection of the given line and the plane is determined without additional constructions.

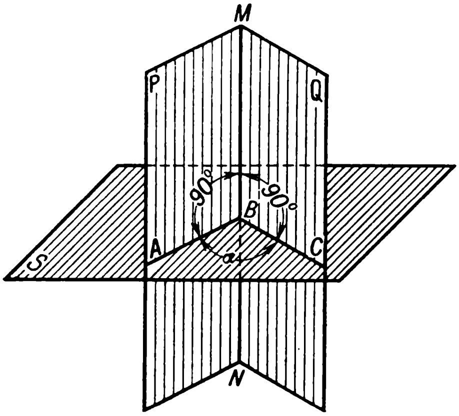

Fig. 199

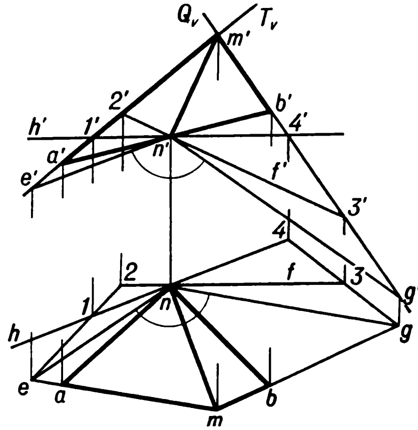

Fig. 200

Two intersecting planes form four dihedral angles. Confining ourselves to considering the angle between *P* and *Q* shown in Fig. 199, let us construct its plane angle for which purpose we intersect the edge *MN* of the dihedral angle by a plane *S* which is perpendicular to *MN*.

The projections of the plane angle are constructed in Fig. 200. The plane *P* is specified by the triangle *AMN*, the plane *Q* by the triangle *BMN*.

(a) A plane *S* passing through the point *N* is constructed perpendicular to *MN* (this plane is specified by its vertical line *NF* and horizontal line *NH*);

(b) the line of intersection of the planes *P* and *S* (the line *EN*) is constructed; since the plane *S* is passed through the point *N* belonging to the plane *P*, we had to find only the point *E* for which purpose an auxiliary plane *T* is taken;

(c) the line of intersection of the planes *Q* and *S* (the line *NG*) is found; here also we had to determine only point *G* (an auxiliary plane *Q*).

The point *N* is the vertex of the required plane angle whose horizontal and vertical projections are represented by the angles *eng* and *e'n'g'*, respectively.

In Figure 195 the projections of the plane angle which measures the dihedral angle formed by the plane *P* with the *H* plane are constructed. Since for obtaining a plane angle we have to pass a plane perpendicular to the edge of the dihedral angle, a plane *S* is drawn perpendicular to the trace P_h to get the angle of inclination of the plane *P* to the *H* plane. Analogously, to obtain the angle between the plane *P* and the *V* plane we would have to pass a plane perpendicular to the trace P_v.

In Figure 195 the vertical projection of the required angle is represented by the angle *n'm'n*, and its horizontal projection coincides with the trace S_h. The magnitude of the angle can be determined by constructing a right-angled triangle whose legs are *n'n* and *mn*.

QUESTIONS TO SECS. 29-31

1. How are the projections of a perpendicular to a plane arranged?

2. What are the relative positions of the horizontal projections of a perpendicular to a plane and of its slope line drawn through the point of intersection of the perpendicular with the plane?

3. How is a plane perpendicular to a given line passed (through a point on the line and through a point outside the line)?

4. How is a perpendicular drawn from a point to an oblique line (with the aid of a plane perpendicular to the line, and by introducing an additional plane of projection into the system V, H)?

5. How are mutually perpendicular planes constructed?

6. In what cases does mutual perpendicularity of one pair of like traces of planes correspond to mutual perpendicularity of the planes themselves?

7. In what case is mutual perpendicularity of planes in the system V, H represented by mutual perpendicularity of their vertical traces? In what case is mutual perpendicularity of planes in the system V, H represented by mutual perpendicularity of their horizontal traces?

8. Are oblique planes mutually perpendicular if their like traces are perpendicular to each other?

9. How is the angle between a straight line and a plane defined and what is the procedure of constructing the projections of this angle in the drawing?

10. How do we proceed in constructing the projections of the plane angle for a given dihedral angle?

CHAPTER 5

THE METHOD OF REPLACING PROJECTION PLANES AND THE METHOD OF REVOLUTION

Sec. 32. Bringing Straight Lines and Plane Figures to Particular Positions Relative to Projection Planes

Straight lines and plane figures specified in particular positions relative to the planes of projection (see Secs. 11 and 19) considerably simplify relevant constructions and solutions of problems, and sometimes permit us to get the answer directly from the given drawing or with the aid of simple constructions.

For example, the determination of the distance of a point A to the horizontal projecting plane (Fig. 201) specified by the triangle BCD is reduced to drawing a perpendicular from the projection a to the projection represented by the line segment bd. The required distance is determined by the line segment ak.

The methods set forth in the present chapter make it possible to pass from oblique positions in which straight lines and plane figures are specified in the system V, H to particular positions in the same system or in an additional one.

This is achieved by:

(1) introducing additional planes of projection so that a straight line or a plane figure, without changing its position in space, is brought to a particular position in a new system of projection planes (*the method of replacing projection planes*);

(2) changing the position of a straight line or a plane figure by revolving it about a certain axis so that a line or a figure is brought to a particular position relative to a fixed system of projection planes (*the method of revolution and its particular case—the coincidence method*).

The introduction of additional planes of projection into the system V, H was considered in Sec. 8, and examples of constructions in additional systems were given in Secs. 13 and 15. Now we are going to consider this in more detail.

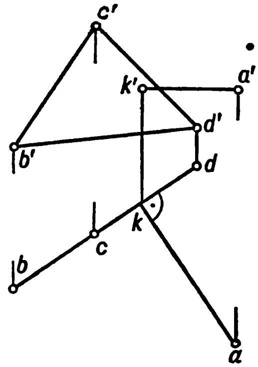

Fig. 201

Sec. 33. The Method of Replacing Projection Planes*

General. The main point of this method consists in that the positions of points, lines, plane figures, and surfaces in space remain unchanged, while the system V, H is supplemented with planes forming (with V or H, or with one another) systems of two mutually perpendicular planes taken for the planes of projection.

Each new system of projection planes is chosen so as to obtain a position most convenient for carrying out the required construction.

In a number of cases for obtaining a system of projection planes suitable for solving the given problem it turns out to be sufficient to introduce only one plane, say a plane S perpendicular to H, or a plane T perpendicular to V; S appearing to be a horizontal projecting plane and T a vertical projecting plane. If the introduction of one plane, S or T, is not sufficient to solve the problem, the basic system of planes should be successively supplemented with new planes: for instance, introducing a plane S perpendicular to the H plane, we get the first new system—S, H, and then, introducing a certain plane T perpendicular to S, we pass to the second new system. In addition, the plane T turns out to be an oblique plane in the principal system V, H. Thus, a successive passage from the system V, H to the system S, T occurs via an intermediate system S, H.

If the planes S and T are yet insufficient to solve the problem completely, we may pass over to a third new system by introducing one more plane perpendicular to T.

When constructing in a new system of projection planes, the rules pertaining to the position of the viewer established for the system of planes V and H (see Sec. 7) must be strictly observed.

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*We use here the usual term 'replacing', though in reality the projection planes V and H are retained, being only supplemented with additional planes of projection.

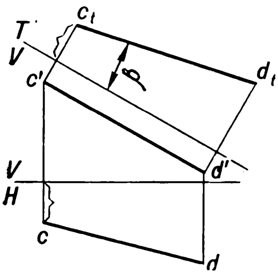

Fig. 202

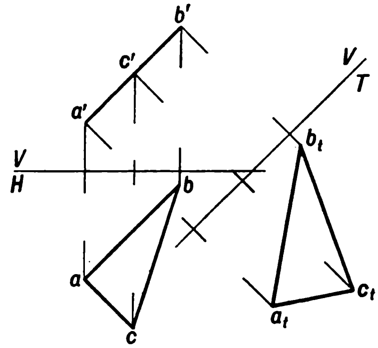

Fig. 203

We shall designate the axis in the form of a common fraction assuming that the stroke lies on this axis. The planes denoted, as usual, by letters will represent the numerator and denominator of the fraction, either letter being put on that side of the axis where the corresponding projections should be located.

Introducing One Additional Plane of Projection into the System *V*, *H*. In most cases the additional plane introduced into the system *V*, *H* as a plane of projection is chosen according to a certain condition to be met in the course of construction. An example is given in Fig. 77: since it was required to determine the true length of the line segment *AB* and the angle between the line *AB* and the *H* plane, the plane *S* was arranged perpendicular to the *H* plane (forming a system *S*, *H*) and parallel to the line segment *AB*.

In Fig. 202 the plane *T* is also chosen to serve quite a definite purpose, i.e. to determine the angle between the line *CD* and the *V* plane. That is why *T* is perpendicular to *V* and at the same time the plane *T* is parallel to the line *CD* (the axis *T*/*V* is parallel to $c'd'$). In addition to the required angle β, the true length of the line segment *CD* is also determined (it is represented by the projection c_td_t).

In the case represented in Fig. 203 the choice of the plane *T* is fully dependent on the task: to determine the true size of the triangle *ABC*. Since in this case the plane defined by the triangle is perpendicular to the *V* plane, to get an untwisted projection of the triangle we have to introduce into the system *V*, *H* an additional plane which would meet the following two conditions: the plane *T* must be perpendicular to the *V* plane (to form a new system *V*, *T*), and the plane *T* must be parallel to the plane containing the triangle *ABC* (which enables the triangle *ABC* to be projected without twisting). The new axis *V*/*T* is drawn parallel to the projection $a'c'b'$. The projection $a_tb_tc_t$ is constructed by laying off from the new axis line segments equal to the distances of the points *a*, *b*, and *c* from the axis *V*/*H*. The true shape of the triangle *ABC* is represented by its new projection $a_tb_tc_t$.

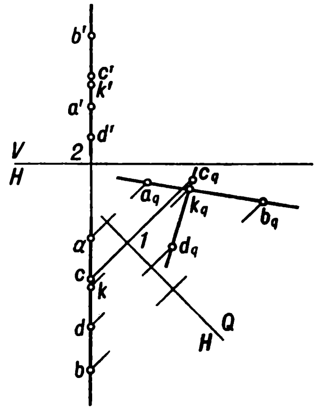

Fig. 204

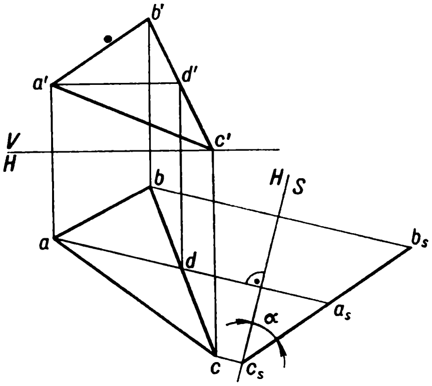

Fig. 205

Figure 204 gives an example of construction in which the plane Q is not specified. Any horizontal-projecting or vertical-projecting or profile plane may be chosen to be the Q plane, provided it is convenient for constructing the required projections. The purpose of construction is to get the projections of the point of intersection of two profile lines AB and CD contained in a common profile plane*. Fig. 204 shows a horizontal projecting plane Q taken as an additional plane of projection.

The relative positions of the new projections a_qb_q and c_qd_q define the relative positions of the given lines: in the given case the lines intersect. The point k_q is the projection of the point of intersection on the plane Q. Using this point, we find the projections k and k'.

The introduction of an additional plane of projection enables us, for instance, to transform a drawing so that an oblique plane specified in the system V, H becomes perpendicular to the additional plane of projection. An example is given in Fig. 205 where an additional plane S is drawn so that the oblique plane defined by the triangle ABC has become perpendicular to the plane S. How is this obtained?

In the triangle ABC a horizontal line AD is drawn. The plane perpendicular to AD is perpendicular to ABC and at the same time to the H plane (since AD is parallel to H). This is satisfied by the plane S, the triangle ABC is projected on it into a line segment b_sc_s. And if an oblique plane is specified

*The fact that straight lines AB and CD intersect follows from the relative positions of the points A and B, C and D.

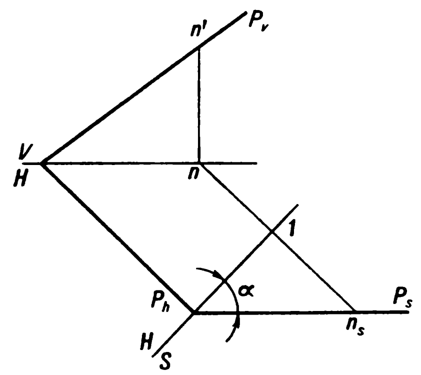

Fig. 206

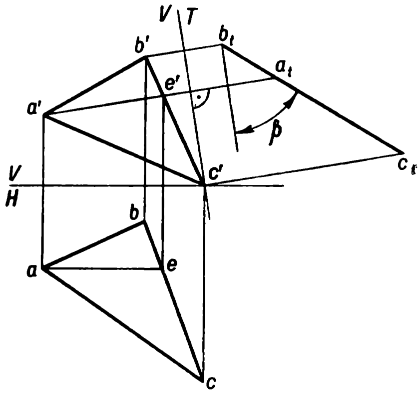

Fig. 207

by its traces (Fig. 206), then the plane S should be passed perpendicular to the trace P_h, i.e. to the line of intersection of the planes P and H. As a result, the plane S will turn out to be perpendicular both to the H plane (i.e. will be an additional plane of projection) and to the plane P. We now have to construct the trace of the plane P on the plane S. Since P is perpendicular to S, the projection of any point of the plane P on the plane S is obtained on the line of intersection of these planes, i.e. on the trace P_s. In Fig. 206 a point N is taken on the trace P_v, its projection n_s is constructed ($n_s1 = n'n$). The trace P_s passes through n_s and also through the point of intersection of the trace P_h with the axis S/H.

The constructions carried out in Figs. 205 and 206 determine the angle α of inclination of the given planes to the H plane. If a plane T (see Fig. 207) is taken perpendicular both to the V plane and to the plane specified by the triangle ABC (for which purpose the axis V/T has to be drawn perpendicular to a vertical line of this plane), then the angle β of inclination of the plane ABC to the V plane will be determined.

Introducing Two Additional Planes of Projection into the System V, H. We are going to consider the introduction of two additional planes of projection into the system V, H using the following example.

Suppose it is required to arrange an oblique line specified in the system V, H perpendicular to the additional plane of projection. Can this be achieved by introducing only one additional plane? No, it is impossible, since such a plane being perpendicular to an oblique line will become an oblique plane in the system V, H, i.e. perpendicular neither to H, nor to V. But this will violate the condition forseen for introducing supplementary planes of projection (see Sec. 8).

Is it possible to avoid this difficulty and still apply the method of replacing the projection planes? Yes, but we have to adhere to the following procedure: to pass over from the system V, H to the system S, H in which S is

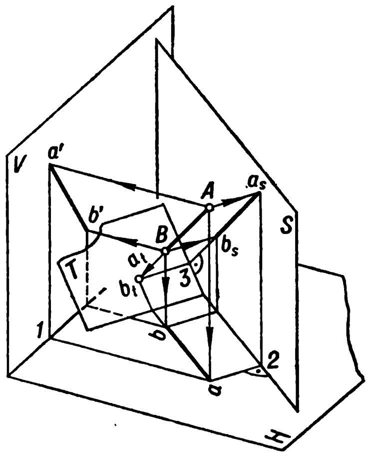

Fig. 208

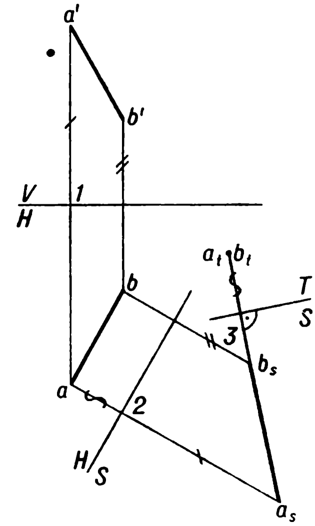

Fig. 209

perpendicular to H and S is parallel to AB, and then to pass over to the system S, T where T is perpendicular to S and T is perpendicular to AB (Fig. 208). The corresponding drawing is given in Fig. 209. The problem is reduced to a successive construction of the projections a_s and a_t of the point A, and b_s and b_t of the point B. The oblique line in the system V, H turns out to be perpendicular to the additional plane of projection T via an intermediate stage of parallelism with respect to the additional plane S. Since the plane S is parallel to the line AB, the points A and B are equidistant from the plane S, this distance being equal, for instance, to the line segment $a2$. Taking the axis S/T perpendicular to a_sb_s (which in space corresponds to perpendicularity of the plane T to the line AB) and laying off the line segment a_t3 equal to $a2$, we get both projections, a_t and b_t, as a single point, i.e. just the thing to be obtained if AB is perpendicular to T.

Figure 210 gives an example of constructing the true size of the triangle ABC. Here also two additional planes of projection (S and T) are introduced, but following such a scheme: S is perpendicular to H and to ABC, and T is perpendicular to S and parallel to ABC. The final stage of construction is reduced to passing the plane T parallel to ABC (since it was required to determine the true size of the triangle ABC); the perpendicularity of the additional plane S to the plane ABC being the intermediate stage. This intermediate stage repeats the construction shown above in Fig. 205. In the concluding stage of the construction shown in Fig. 210 the axis S/T is parallel to the projection $c_sa_sb_s'$, i.e. the plane T is drawn parallel to the plane ABC which leads to the determination of the true size of the triangle represented by the projection $a_tb_tc_t$.

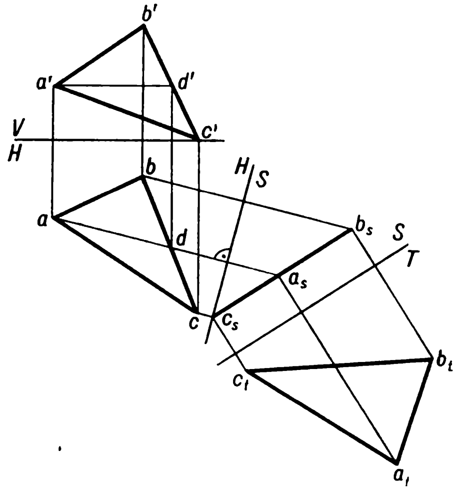

Fig. 210

Thus, to obtain parallelism of the planes *ABC* and *T* in this example we had first to arrange the triangle *ABC* and the plane *S* in mutually perpendicular positions. And vice versa, to get perpendicularity (*AB* to *T*) in the example given in Fig. 209 we had first to obtain mutually parallel positions of *AB* and *S*.

QUESTIONS TO SECS. 32-33

1. What methods of transformation of drawing are considered in Chapter 5?

2. What is the difference between these methods?

3. What does the method of replacing projection planes consist in?

4. What position in the system *V*, *H* must occupy the plane of projection *S* introduced to form the system *S*, *H*?

5. What position will the projection plane *T* occupy in the system *V*, *H* when the latter is transformed from *V*, *H* via *S*, *H* to *S*, *T*?

6. How do we find the true length of a line segment and the true size of the angles formed by this line with the planes *V* and *H* by introducing supplementary planes of projection?

7. How many additional planes should be introduced into the system *V*, *H* to determine the true size of a figure whose plane is perpendicular to *H* or to *V*?

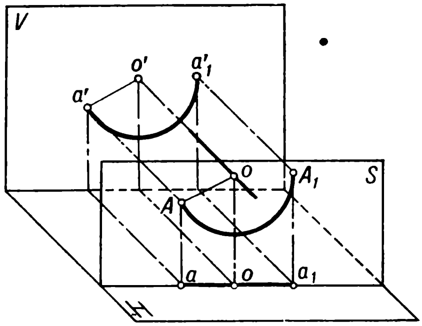

Fig. 211

8. How many planes and in what succession must be introduced into the system V, H to bring a given oblique plane to the position in which it is perpendicular to the additional planes of projection?

9. The same question but with respect to obtaining the true size of a figure contained in an oblique plane.

Sec. 34. The Method of Revolution Characterized

When revolving about a certain fixed line (called the *axis of revolution*) each point of the revolved figure displaces in the plane perpendicular to the axis of revolution (called the *plane of revolution*). A point describes a circle whose centre lies at the point of intersection of the axis of revolution and the plane in which the point revolves (the *centre of revolution*), the radius being equal to the distance of the revolved point from the centre (the *radius of revolution*). If a point belonging to a given system lies on the axis of revolution, then during rotation of the system this point is considered to be fixed.

The axis of revolution may be specified or chosen; in the latter case it is advantageous to arrange the axis perpendicular to one of the projection planes, since this simplifies relevant constructions.

Indeed, if the axis of revolution is perpendicular, say, to the V plane, then the plane in which the point revolves is parallel to the V plane. Hence, the path described by the point is projected on the V plane without distortion, and on the H plane into a line segment (Fig. 211).

Sec. 35. Revolution of a Point, a Line Segment and a Plane about an Axis Perpendicular to a Projection Plane

Revolution about a Given Axis.

1. Let a point A revolve about an axis perpendicular to the H plane (Fig. 212). Through the point A a plane T is passed perpendicular to the axis of revolution and, consequently, parallel to the H plane. Revolving in the plane T, the point A describes a circle of radius R which is represented by the length of the perpendicular dropped from the point A onto the axis. The circle described by the point A in space is projected on the H plane without distortion. Since the plane T is perpendicular to the V plane, the projections of the points of the circle on the V plane will be located on T_v, i.e. on the line perpendicular to the vertical projection of the axis of revolution. In the drawing given in Fig. 212 (right), the circle described by the point A when the latter is revolved about the axis, is projected on the H plane without distortion. A circle of radius $R = oa$ is described from point o as centre; on the V plane this circle is represented by a line segment equal to $2R$.

Figure 213 illustrates rotation of a point A about an axis perpendicular to the V plane. The circle described by the point A is projected on the V plane without distortion. From point o' as centre, a circle of radius $R = oa$ is drawn which is represented on the H plane by a line segment equal to $2R$.

From Figures 212 and 213 it is obvious that *when rotating a point about an axis perpendicular to a projection plane one of the projections of the rotated point displaces in a straight line perpendicular to the projection of the axis of revolution.*

Figure 214 shows how a point A is rotated anticlockwise through an angle α about an axis passing through the point O perpendicular to the V plane. From point o' as centre an arc of radius $o'a'$ is described corresponding to the angle α and the direction of rotation. Point a_1' is the new position of the vertical projection of the point A.

2. Now we are going to consider the rotation of a line segment about a given axis. The line segment AB (Fig. 215) is turned to the position A_1B_1. Obviously, the problem is reduced to the rotation of points A and B through a given angle α in a given direction. The paths along which the vertical projections of these points are displaced are indicated by straight lines drawn through a' and b' perpendicular to the vertical projection of the axis of revolution.

The new position of the horizontal projection of the point A (point a_1) is obtained by rotating the radius oa through a given angle α. To find the point b_1 (the position of the horizontal projection of the point B after the rotation) a circular arc of radius ob is described and in this arc a chord bb_1 is laid off equal to the chord *1-2*. This corresponds to the rotation of the point B through the same angle α.

Then from the points a_1 and b_1 the line of recall are drawn to intersect

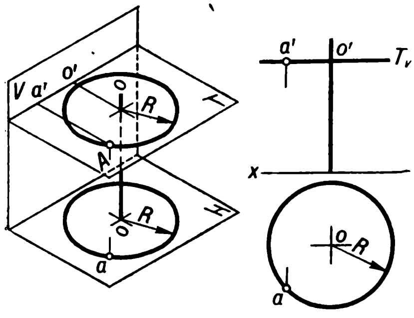

Fig. 212

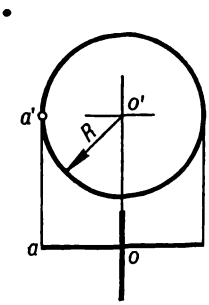

Fig. 213

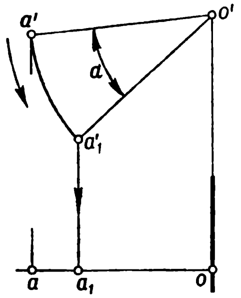

Fig. 214

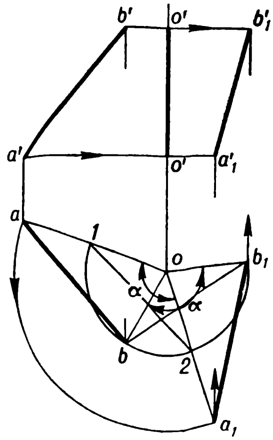

Fig. 215

the directions of displacement of the vertical projections; as a result, the projections a_1' and b_1' are obtained.

The line segments between the points a_1' and b_1', and between the points a_1 and b_1 determine the new positions of the vertical and horizontal projections of the line segment after its rotation to the position A_1B_1.

Let us now consider the triangles abo and a_1b_1o (see Fig. 215). The sides bo and ao of the triangle abo are respectively equal (as radii of one and the same circle) to the sides b_1o and a_1o of the triangle a_1b_1o, and the angles between these sides are also equal. Hence, the triangles are congruent, and ab is equal to a_1b_1, i.e. *the length of the horizontal projection of a line segment revolved about an axis perpendicular to the H plane remains unchanged.* Obviously, a similar conclusion is true for *the vertical projection of a line segment revolved about an axis perpendicular to the V plane.*

In the congruent triangles abo and a_1b_1o (Fig. 215) the altitudes drawn, for instance, from point o to ab and a_1b_1 will also be equal.

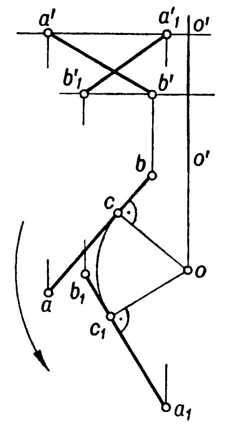

Fig. 216

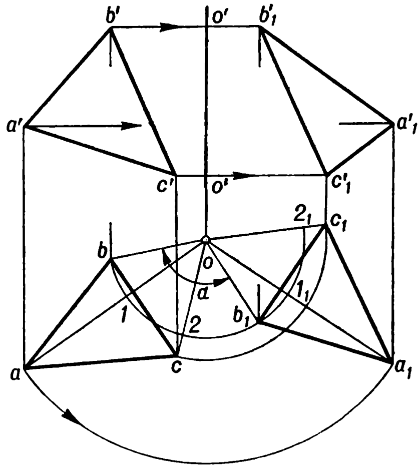

Fig. 217

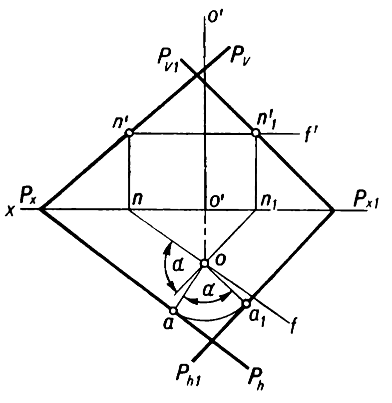

Fig. 218

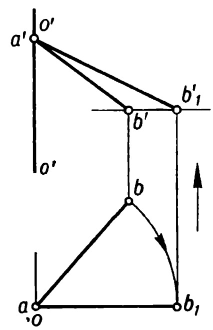

Fig. 219

The conclusions drawn enable us to establish the following method of constructing the new projections of a line segment revolved about an axis through a given angle (Fig. 216). Through the point o we draw a straight line perpendicular to ab, and rotate the point c (the intersection of the perpendicular with ab) through a given angle. Drawing through the point c_1 (the new position of the point c) a straight line perpendicular to the radius oc_1, we obtain the direction of the new position of the horizontal projection of the line segment. Since the length of the line segments ca and cb remains unchanged, we find the new position a_1b_1 by laying off from the point c_1 the line segments $c_1a_1 = ca$ and $c_1b_1 = cb$. The location of the new position of the vertical projection $a_1'b_1'$ remains unchanged.

The above method may be used not only for revolving a line segment through a given angle, but also for determining the angle through which a given segment must be revolved to occupy a certain required position (for instance, for arranging it parallel to the V plane).

3. The revolution of a plane about a given axis is reduced to rotating the points and straight lines belonging to it.

An example is given in Fig. 217: the triangle ABC defining the plane is revolved to occupy the position $A_1B_1C_1$ according to the given angle α and direction indicated by an arrow. The construction is similar to that shown in Fig. 215 where two points A and B were revolved. But here three points (the vertices A, B and C) are revolved, and, hence, the whole figure. The triangles abc and $a_1b_1c_1$ are congruent by construction: with the axis perpendicular to the H plane the horizontal projection remains unchanged. This corresponds to that the angle of inclination of the plane ABC to the H plane remains unchanged if the axis of revolution is perpendicular to the H plane. Obviously, when the figure is revolved about an axis perpendicular to the V plane, the angle of inclination of the revolved plane to the V plane and the vertical projections remain unchanged.

When revolving a plane represented by its traces, we usually rotate one of the traces and a horizontal (vertical) line of the plane. An example is given in Fig. 218 where an oblique plane P is revolved through an angle α about an axis perpendicular to the H plane. On the trace P_h a point is taken nearest to the axis of revolution, i.e. a point a (oa is perpendicular to P_h) similar to the point c taken in Fig. 216. The point a is then revolved through an angle α. Through the point a_1 thus obtained we draw a straight line perpendicular to oa_1 which is the horizontal trace of the plane in its new position.

To find the vertical trace of the plane after its rotation it is sufficient to find, in addition to the point P_{x1} determined on the x-axis, one more point belonging to the trace. In the plane P we take a horizontal line nf, $n'f'$ which intersects the axis of revolution (nf passes through the horizontal projection of the axis of revolution). Of course, we might take a horizontal line which does not intersect the axis of revolution. Since the horizontal line in its new position remains parallel to its horizontal trace, we have to draw through o a straight line parallel to P_{h1} thus obtaining the new position of the horizontal projection of the horizontal line. The vertical projection will not change its direction, and therefore it is easy to find the new vertical trace of the horizontal line—point n_1'. Now we can construct the vertical trace of the plane (P_{v1}).

Revolution About a Chosen Axis. In a number of cases the axis of revolution may be chosen. And if the axis of revolution is chosen as passing through one of the end-points of a line segment, then the construction will become simplified, since the point through which the axis passes will be 'fixed'. In this case, to revolve the line segment we have to construct the new positions of the projections only of one point (the other end-point of the segment).

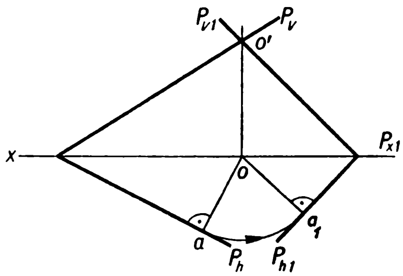

Fig. 220

Figure 219 illustrates the case when for revolving the line segment AB the axis of revolution is chosen to be perpendicular to the plane H and passing through the point A. When revolving about such an axis, we can, for instance, position the line segment parallel to the V plane. This position is just shown in Fig. 219. The horizontal projection of the line segment in its new position is perpendicular to the line of recall aa'. Finding the point b_1' and constructing the line segment $a'b_1'$, we get the vertical projection of the line segment AB in its new position. The projection $a'b_1'$ represents the true length of the line segment AB, the angle $a'b_1'b'$ being equal to the angle between the line AB and the H plane.

If it is required to determine the angle of inclination of an oblique line to the V plane, then we have to draw the axis of revolution perpendicular to the V plane and to revolve the line so that in its new position it is parallel to the H plane. The relevant construction is left to the reader.

If for revolving a plane represented by its traces we may choose the axis of revolution, then it is advisable to arrange it in a projection plane—this will simplify the constructions involved. An example is given in Fig. 220. Suppose the axis of revolution must be perpendicular to the H plane. If it is taken in the V plane, then on the trace P_v there appears a 'fixed' point O (at the intersection with the axis of revolution). After the plane is revolved the vertical trace must pass through this point. Consequently, on finding the position of the horizontal trace (P_{h1}) after the revolution, we have to draw the trace P_{v1} through the points P_{x1} and o'. The comparison with Fig. 218 shows that the simplification consists in avoiding a horizontal line. It would be necessary if the point P_{x1} 'left' the limits of the drawing; but in similar case in Fig. 218 we would have to take two auxiliary lines.

Figure 221 shows how an oblique plane becomes a horizontal projecting plane after the revolution and the angle of inclination of the plane P to the V plane is thus determined. If we take an axis of revolution perpendicular to the H plane, then it is possible to bring the plane P to the position of a vertical projecting plane, and thus determine the angle of inclination of the plane P to the H plane.

Comparing the positions occupied by the plane before and after the

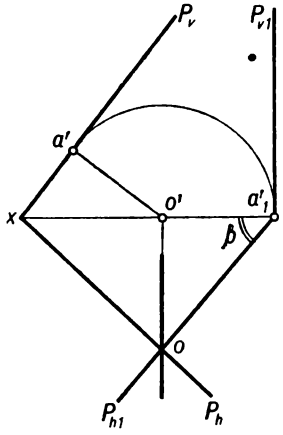

Fig. 221

rotation, we note that the angle formed by the traces P_v and P_h in the drawing is changed in general.

If we imagine a circular cone with the vertex at the point O and with the base contained in the H plane in Fig. 220 and in the V plane in Fig. 221, and a plane P tangent to the cone, then the revolution of the plane P about the axis of revolution coinciding with the axis of the cone represents a kind of 'rolling' the cone by the tangent plane.

QUESTIONS TO SECS. 34-35

1. What does the method of revolution consist in?

2. What is the plane of revolution of a point and how is it positioned with respect to the axis of revolution?

3. What is the centre of revolution of a point when the latter is revolved about a certain axis?

4. What is the radius of revolution of a point?

In the following questions revolution is meant about an axis perpendicular to a projection plane.

5. How are the projections of a point displaced?

6. What projection of a line segment does not change its length?

7. How do we revolve a plane: (a) not represented by its traces, (b) represented by the traces?

8. In what case does the revolution preserve the angle of inclination of a straight line with respect to the plane: (a) H, (b) V.

9. The same question with reference to the W plane.

10. Is it possible to determine by revolution the true length of a line segment and the angle of its inclination to the planes V and H?

11. Is it possible by revolving a plane to determine the angles of its inclination to the planes V and H?

12. In what advantageous position can we place the axis of revolution when rotating (1) a line segment, (2) a plane represented by its traces?

Sec. 36. Applying the Method of Revolution Without Indicating in the Drawing the Axes of Revolution Perpendicular to the *V* or *H* Plane

In Section 35 we saw that if a line segment or a plane figure is revolved about an axis perpendicular to a projection plane then its projection onto this plane preserves its shape and size, the position of this projection with respect to the axis of projection being the only thing which undergoes a change. As far as the other projection is concerned, the one on the plane parallel to the axis of revolution, we see that all points belonging to this projection (except for the projections of points situated on the axis of revolution) displace in straight lines parallel to the axis of projection, and the projection in general changes its shape and size. Taking advantage of these properties, we can apply the method of revolution without representing the axis of revolution and without specifying the length of the radius of revolution; it is sufficient only to displace one of the projections of the figure under consideration into the required position without changing its shape and size, and then to construct the other projection as described above.

For instance, it is required to rotate the segment AB of an oblique line (Fig. 222) so as to position it perpendicular to the H plane. We begin with the revolution about the axis perpendicular to the H plane until the position parallel to the V plane is achieved without indicating this axis in the drawing. Since this revolution leaves the horizontal projection of the line segment unchanged, we take the projection a_1b_1 equal to ab and position it parallel to the x-axis which corresponds to parallelism of the line segment to the V plane.

On finding the corresponding vertical projection of the line segment ($a_1'b_1'$) we carry out a second revolution, this time about an axis perpendicular to the V plane until the required position is obtained in which AB is perpendicular to the H plane. This axis is not represented in the drawing either. We place the projection $a_2'b_2'$ equal to $a_1'b_1'$ perpendicular to the x-axis. The horizontal projection of the segment is represented by a point designated by two letters—a_2b_2.

Thus, the performed operations correspond to the revolutions about axes perpendicular to projection planes, but these axes are not shown in the drawing. Of course, they can be found. For instance, if we draw two straight lines—one through the points a and a_1, the other through b and b_1—and erect perpendiculars at the mid-points of the line segments aa_1 and bb_1, then the obtained point of intersection of these perpendiculars will be the horizontal projection of the axis of revolution perpendicular to the H plane. But, as we see, there is no need to indicate them.

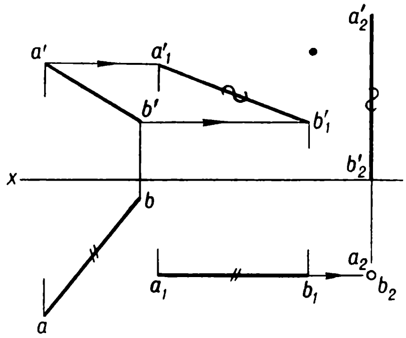

Fig. 222

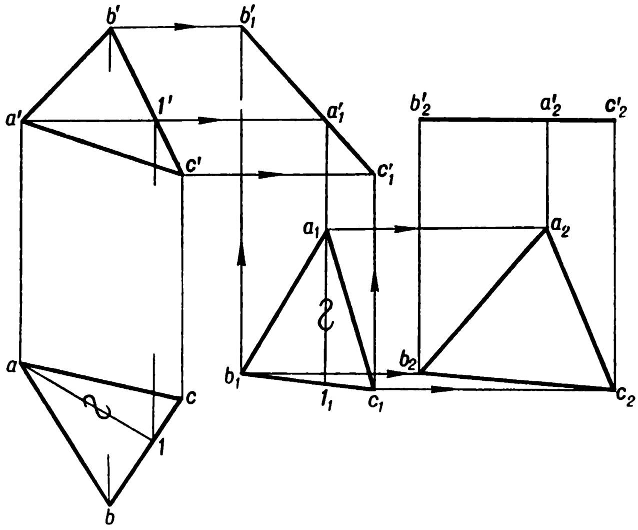

Fig. 223

Figure 223 demonstrates two stages of revolving the triangle *ABC* contained in an oblique plane for determining the true size of this triangle. Indeed, in its latter position it is parallel to the *H* plane and, consequently, the projection $a_2b_2c_2$ represents the true size of the triangle. But to bring it to such a position we first have to revolve the oblique plane containing the triangle so as to arrange it perpendicular to the *V* plane. To this end we have to take a horizontal line in the triangle *ABC* and revolve it until it is perpendicular to the *V* plane; then the triangle containing this horizontal line will also become perpendicular to the *V* plane. Since the construction is carried out without indicating the axes of revolution, we arrange the pro-

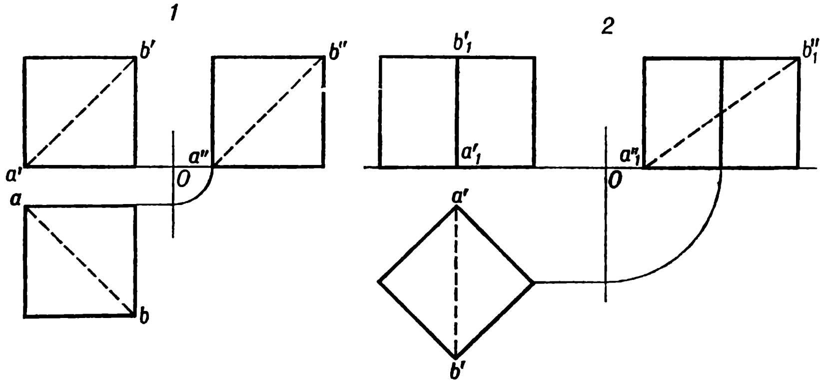

Fig. 224

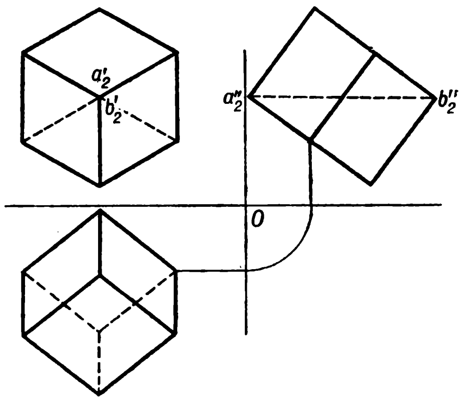

Fig. 225

jection $a_1b_1c_1$ arbitrarily, but so that the horizontal line is perpendicular to the V plane; for this purpose we direct the projection of the horizontal line a_1l_1 parallel, say, to the line of recall $a'a$ (the drawing contains no axis of projection). This revolution is understood as being accomplished about an axis perpendicular to the H plane; therefore the horizontal projection of the triangle preserves its shape and size ($a_1b_1c_1 = abc$), changing only its position. During such a revolution the points A, B and C displace in planes parallel to the H plane, the projections b_1', a_1' and c_1' being situated on the horizontal lines of recall $a'a_1'$, $b'b_1'$, and $c'c_1'$.

The second revolution which brings the triangle to a position parallel to the H plane is meant as being accomplished about an axis perpendicular to the V plane. During this revolution the vertical projection retains its shape and size, the points A_1, B_1, and C_1 obtained during the second stage of revolution now displace in planes parallel to the V plane; and the projections a_2, b_2, and c_2 are found on the horizontal lines of recall with the points a_1, b_1, c_1.

The projection $a_2b_2c_2$ represents the true shape and the size of the triangle ABC.

This method rather simplifies the constructions involved and avoids superimposed projections, but the drawing occupies a greater area*.

One more example of revolution without showing the axes is given in Figs. 224 and 225. Here a cube is successively revolved until it is brought to a position in which the diagonal AB is arranged perpendicular to the V plane.

First the cube is revolved about an axis perpendicular to the H plane until it is positioned so that the diagonal AB is placed in a profile plane (Fig. 224).

The cube is then brought to a third position in which the diagonal AB turns out to be perpendicular to the V plane (Fig. 225). This is obtained by revolving the cube about an axis perpendicular to the W plane**.

Sec. 37. Revolution of a Point, a Line Segment, and a Plane about an Axis Parallel to a Projection Plane, and about a Trace of a Plane

Revolution of a Plane Figure about Its Horizontal Line. To determine the true shape and dimensions of a plane figure, the latter may be revolved about a *horizontal line* belonging to it with the purpose to arrange the figure parallel to the H plane.

Let us first consider the revolution of a point (Fig. 226). The point B revolves about a horizontally arranged axis On' describing a circular arc contained in the plane S. This plane is perpendicular to the axis of revolution and, hence, is a horizontal projecting plane. Therefore, the horizontal projection of the circle described by the point B must lie on S_h.

If the radius OB occupies a position parallel to the H plane, then the projection ob_1 will turn out to be equal to OB_1, i.e. equal to the true length of the radius OB.

Now we are going to examine Fig. 227, which illustrates the revolution of a triangle ABC about a horizontal line AD taken as the axis of revolution. The point A located on the axis of revolution will become fixed. Consequently, to represent the horizontal projection of the triangle after the revolution we have to find the positions of the projections of two other of its vertices. Dropping a perpendicular from point b onto ad, we find the horizontal projection of the centre of revolution, i.e. the point o, and the horizontal projection of the radius of revolution of the point B (the line segment ob), and then determine the vertical projection of the centre of revolution

*This method is sometimes called 'the method of plane-parallel displacement'.

**The projection of the cube thus obtained on the V plane (Fig. 225) coincides with its representation in orthogonal isometry.

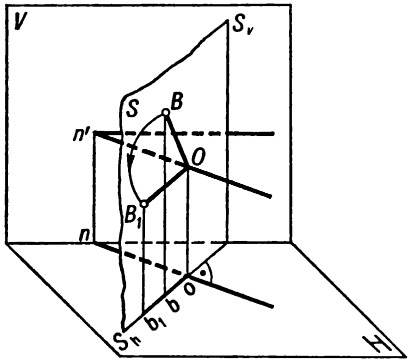

Fig. 226

(the point o' and the vertical projection of the radius of revolution of the point B—the line segment $o'b'$. Now we have to determine the true length of the radius of revolution of the point B. This is done with the aid of the method described in Sec. 13, i.e. by constructing a right triangle. Using the line segments ob and $b\bar{B} = b'1'$ as the legs, we construct a right triangle $ob\bar{B}$; its hypotenuse is equal to the radius of revolution of the point B.

Now we can find the position of the point b_1, and then of the point c_1. Here we may not determine the radius of revolution of the point C, but find the position of the point c_1 as the intersection of two straight lines, one of which is a perpendicular drawn from the point c to the line ad, the other passing through the found point b_1 and point d (the horizontal projection of the point D belonging to the side BC and situated on the axis of revolution).

The projection ab_1c_1 represents the true size of the triangle ABC, since after the revolution the plane containing the triangle is parallel to the H plane. As far as the vertical projection of the triangle is concerned, it coincides with the vertical projection of the horizontal line, i.e. represents a straight line.

Figure 227 represents the construction for the case when a horizontal line is drawn outside the projections of a triangle. This enables us to avoid superimposed projections, but the drawing occupies a greater area.

If it is required to revolve a plane figure to a position parallel to the V plane, then a *vertical line* should be chosen for the axis of revolution.

We would like to draw reader's attention to the fact that in the construction shown in Fig. 226 we do not see the projection $o'b'$ of the radius of revolution of the point B. Obviously, a proper understanding of the construction in question makes it possible to get rid of this projection. An example is given in Fig. 228 showing how a plane specified by a point K and a line AB is revolved to a position parallel to the H plane. The plane is revolved about a horizontal line KD which is drawn through the point K which is

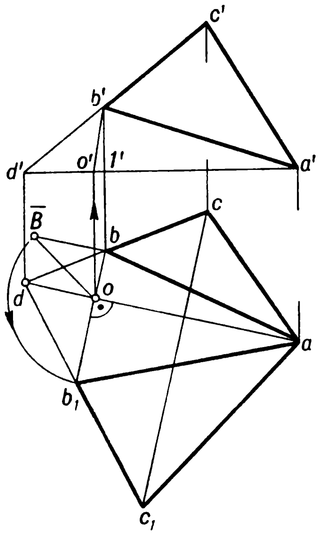

Fig. 227

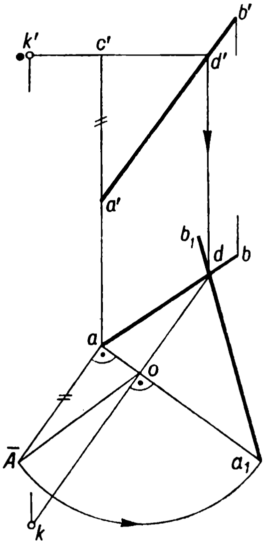

Fig. 228

thus becomes 'fixed'. What is left is to rotate the line AB about KD, or strictly speaking, only the point A, since the point D on the line AB is also 'fixed': it belongs to the axis of revolution. Drawing ao perpendicular to kd, i.e. marking the position of the horizontal trace of the horizontal projecting plane in which the point A revolves, we obtain the horizontal projection of the centre of revolution of the point A (point o) and the horizontal projection of the radius of revolution of the point A (line segment oa). We now find the true length of the radius of revolution R_A as the hypotenuse of the triangle $oa\bar{A}$ in which the leg $a\bar{A}$ is equal to $a'c'$. On finding the point a_1 (the horizontal projection of the point A after the revolution), we draw a_1b_1, i.e. the horizontal projection of the line AB after the revolution, using the point d. Thus, we have done without the vertical projections of the centre and radius of revolution.

Revolution of a Plane about Its Trace. This is a special case of the method of revolution of a plane when one of its traces serves as the axis of revolution. As a result of such revolution, the given plane is brought into coincidence with one of the projection planes. In the latter plane we get a true size representation of the line segments and figures contained in the plane brought to coincidence. This special case of the method of revolution is called the coincidence method.

Obviously, this construction is analogous to the revolution of a plane about its horizontal or vertical line until it is parallel to the corresponding plane of projection: the horizontal trace of the plane may be considered as its 'zero' horizontal line, and the vertical trace as its 'zero' vertical line.

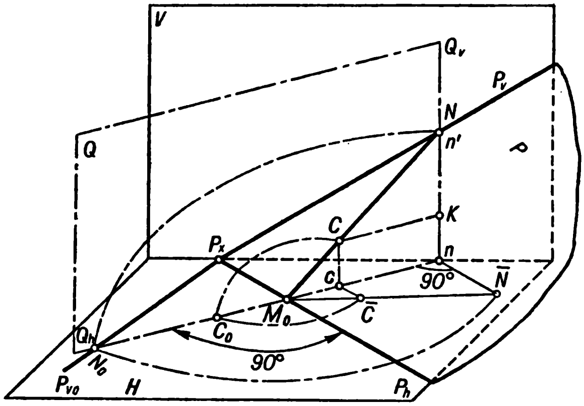

Fig. 229

Figure 229 shows constructions that result from coincidence of an oblique plane P with the H plane, the revolution being carried out about P_h in the direction from the V plane toward the viewer.

Two intersecting lines contained in the plane P will turn out to be coincident with the H plane: the trace P_h and the straight line P_{v0} which represents the trace P_v coincident with the H plane.

The trace P_h, as the axis of revolution, does not change its position; the point of intersection of the traces also preserves its position, and therefore, if it were required to indicate the coincident position of the trace P_v, it would be sufficient to find one more point of this trace (besides the point P_x) in the position coincident with the H plane. We are going to find the coincident position of a point N lying on the trace P_v. This point will describe an arc of a circle in the Q plane perpendicular to the axis of revolution; the centre of this arc lies at the point M_0 of intersection of the plane Q with the trace P_h. Describing from the point M_0 an arc of radius M_0N in the plane Q, we get a point N_0 on the H plane as the point of intersection of this arc with Q_h. Drawing a straight line through P_x and N_0, we obtain P_{v0}. Since the line segment P_xN preserves its length during the revolution of the plane, obviously, we may obtain N_0 as the intersection of Q_h with the arc of radius P_xN described in the plane H.

In the drawing (Fig. 230) an arbitrary point N is taken on the trace P_v (it coincides with its projection n') and through its projection n a straight line nM_0 is drawn perpendicular to the axis of revolution, i.e. to the trace P_h. On this line the point N must lie after it is brought in coincidence with the H plane at a distance from the point M_0 equal to the radius of revolution of the point N, or at a distance P_xn' from the point P_x. The length of the radius of revolution can be determined as the hypotenuse of a right-angled triangle with the legs M_0n and $n\bar{N}$ ($n\bar{N} = nn'$). Describing from the point M_0 an arc of radius $M_0\bar{N}$, or from the point P_x an arc of radius P_xn', we get on the

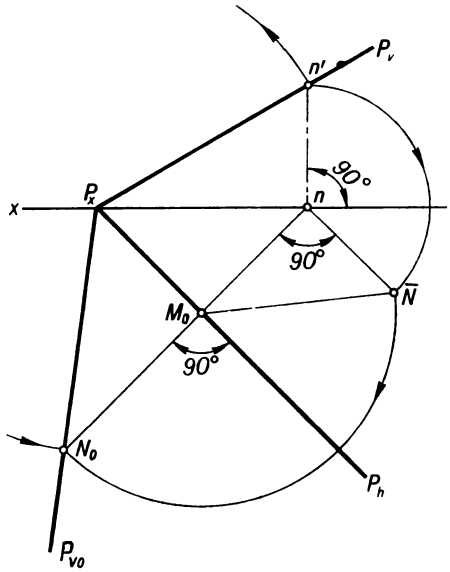

Fig. 230

line nM_0 a point N_0 which is the coincident position of the point N with the H plane. Drawing a straight line through the points P_x and N_0, we obtain the coincident position of the trace P_v (the line P_{v0}).

Let us return to Fig. 229 and consider the coincidence of the point C with the H plane.

The coincident position of the point C with the H plane is found in Fig. 231 (left). Through the point c a straight line cM_0 is drawn perpendicular to P_h. The radius of revolution $M_0\bar{C}$ is determined as the hypotenuse of a right triangle whose one leg is M_0c, its other leg being $c\bar{C} = c'1$. From the point M_0 as centre we describe an arc of radius $M_0\bar{C}$ to intersect the extended line cM_0 at point C_0. The latter is just the coincident position of the point C in the H plane.

This construction may also be accomplished in the way shown in Fig. 231 (right). We first determine the position of the point C in the plane P by means of a vertical line and draw a straight line cM_0 perpendicular to P_h; and then intersect this line with an arc described from the point l as centre whose radius is equal to the line segment $c'l'$, i.e. to the true length of the line segment CL in the plane P. On coincidence this length remains unchanged: $c_0l = CL$.

If a line segment is given in a plane, then finding the coincident positions of the end-points of this segment, we get the *true length of the line segment.*

As is known, any horizontal line taken in the plane P is parallel to P_h, and any vertical line is parallel to P_v. Therefore, if there is a need to find the

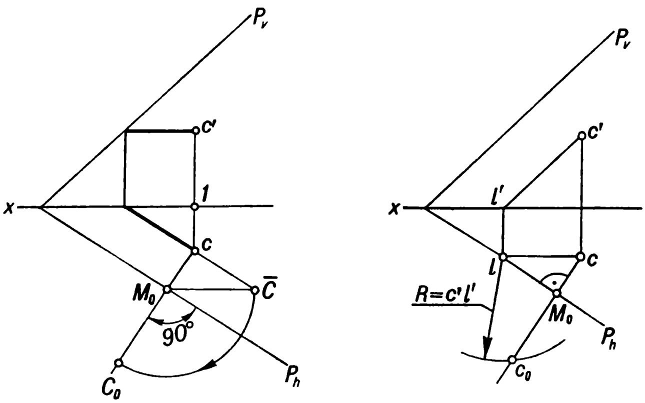

Fig. 231

coincident position of a horizontal or a vertical line, then it is sufficient to find the coincident position of its trace and to draw through it a straight line parallel respectively to P_h or P_{v0} (if the plane P is brought in coincidence with the H plane).

We shall take an advantage of this method for carrying out an inverse construction. Let there be given a point C_0 which is the coincident position of the point C with the H plane; it is required to find the projections of the point C if it must lie in the plane P specified by its traces (see also Fig. 229).

When the point C_0 is 'elevated into space', its horizontal projection (point c) displaces in a straight line C_0n (Fig. 232) perpendicular to P_h, i.e. in the trace Q_h of the plane of revolution Q. The point C in space must lie on the line of intersection of the plane P with the plane of revolution (see Fig. 229) at a distance M_0C_0 from the point M_0.

Let us construct on the H plane a right triangle $M_0n\bar{N}$ in which the side $n\bar{N}$ is equal to $n'n$ (Fig. 232), and which is, consequently, congruent to the triangle Mnn' in space.

Laying off on the hypotenuse $M_0\bar{N}$ a line segment M_0C_0 (the radius of revolution), we get the point $\bar{C}$. Drawing through it a straight line perpendicular to M_0n, we obtain the required position of the horizontal projection of the point C (point c).

The point c' must lie on the perpendicular drawn from the point c to the x-axis at a distance $c'1$ equal to $c\bar{C}$.

If it is required to 'elevate into space' a line segment, then in the general case we have to elevate two of its points in the above-mentioned way, or using a so-called 'fixed' point. This is shown in Fig. 233 where it was required to 'elevate into space' (that is on the plane P) a line segment AB specified by its coincident with the H plane position (A_0B_0). The construction is somewhat complicated by that the point of intersection of the traces P_v and P_h is considered to be inaccessible.

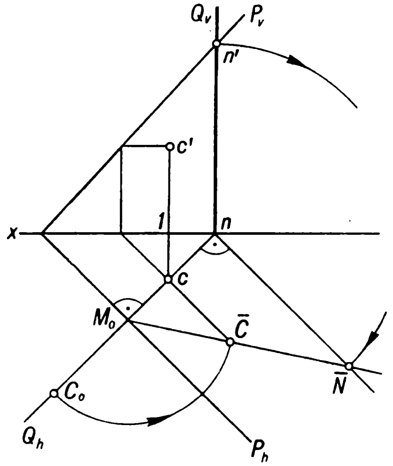

Fig. 232

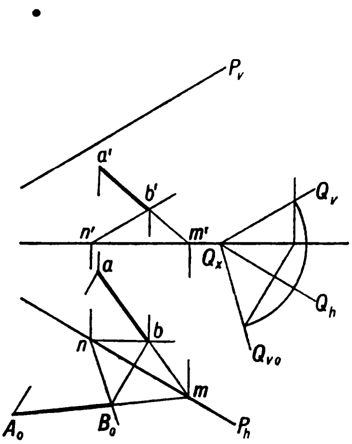

Fig. 233

An auxiliary plane Q is constructed parallel to P and the trace $Q_{\bar{v}}$ is found coincident with the H plane. Since Q is parallel to P, then Q_{v0} determines the direction of the vertical lines of both Q and P in the position coincident with the H plane. Therefore, drawing B_0n parallel to Q_{v0}, we get the vertical line of the plane P on which point B is located in space; as is obvious, this line is obtained in the position coincident with the H plane. We now construct the projections of this vertical line and find on them the projections b and b'. If we extend the line A_0B_0 to intersect the trace P_h at point m, then the horizontal projection ab will be found on the line passing through this 'fixed' point m and the constructed projection b. The projection $a'b'$ is obtained on the line passing through the points m' and b'.

We have considered the coincidence of a plane with the horizontal plane of projection, revolving the plane about the horizontal trace. If it is required to bring it in coincidence with the vertical plane of projection, we have to revolve the plane about its vertical trace.

If a horizontal projecting plane is revolved about its vertical trace until it is coincident with the V plane, then after the revolution the horizontal trace of the plane will be situated on the axis of projection. Likewise, if a vertical projecting plane is revolved about its horizontal trace until it is brought into coincidence with the H plane, then the vertical trace of the plane will be found on the axis of projection.

Figure 234 represents a plane with an obtuse angle between the traces Q_v and Q_h brought into coincidence with the H plane when revolved 'towards the viewer' and in the reverse direction.

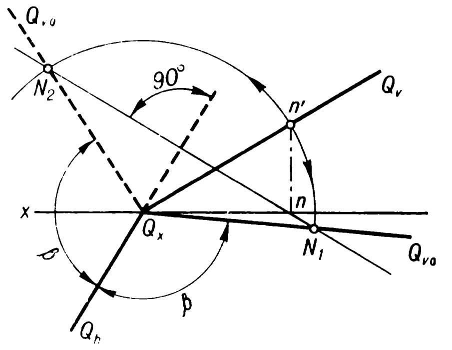

Fig. 234

QUESTIONS TO SECS. 36-37

1. Is it possible to show in the drawing the revolution of a straight line about an axis perpendicular to H or V without representing the axis itself? What is this method based on?

2. What term is sometimes used for the method of revolution without indicating the axis?

3. How is the plane of revolution of a point arranged if the axis of revolution of the latter is only parallel to H or V, but is perpendicular neither to H, nor to V? Why in this case is it necessary to determine the true length of the radius of revolution?

4. What is the test for verifying whether a plane specified by a horizontal line and a point has reached the horizontal position when revolved about this horizontal line, and where is the vertical projection of the point obtained after the revolution?

5. What is understood under term 'the coincidence method'?

6. What is implied by 'elevation into space'?

Sec. 38. Examples of Solving Problems Using the Method of Replacing Projection Planes and the Method of Revolution

1. Construct the projections of the point of intersection of two profile lines contained in a common profile plane.

The solution is given in Fig. 204. Here the method of replacing projection planes is applied. To get the projection k' lay off the line segment $k'2$ equal to the found segment k_q1.

2. Construct an additional plane of projection so that an oblique line turns out to be perpendicular to this plane.

The solution is given in Fig. 209 where two additional planes are succes-

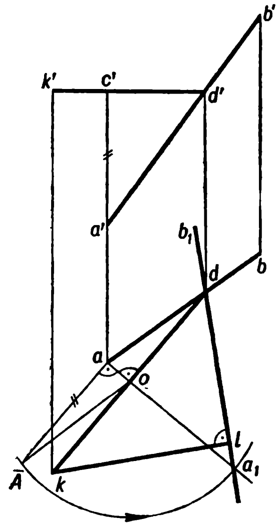

Fig. 235

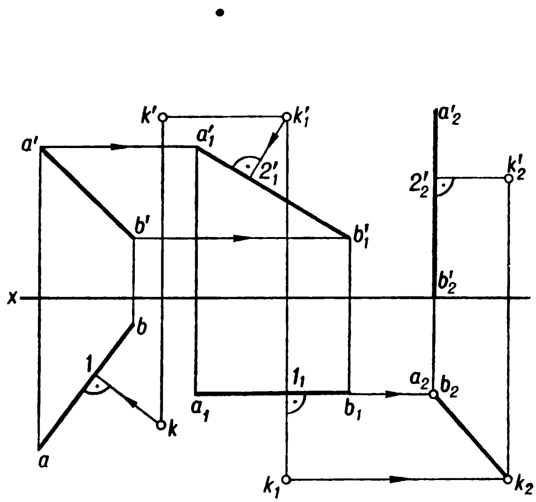

Fig. 236

sively introduced. The line segment *AB* is perpendicular to the second additional plane of projection *T*.

3. Revolve an oblique line to arrange it perpendicular to the *H* plane.

The solution is given in Fig. 222. Two successive revolutions are applied. After the second revolution the line segment *AB* becomes perpendicular to the *H* plane.

4. Determine the true length of a segment of an oblique line and the true size of the angles of its inclination to the projection planes *V* and *H*.

Figure 202 shows the solution by the method of replacing projection planes. An additional plane *T* is introduced which is perpendicular to *V* and parallel to the given segment *CD*, thus determining the length of the line segment and the angle with the *V* plane.

Figure 219 shows the solution by the method of revolution. The line segment *AB* is revolved about the axis drawn through the point *A* to occupy the position parallel to the *V* plane, thus determining the length of the line segment and the angle with the *H* plane.

5. Determine the distance from a point to a straight line.

Let us examine Fig. 228. It shows the revolution of the plane defined by a point *K* and a straight line *AB* about the horizontal line *KD* of this plane. The revolution brings the plane to a position parallel to the *H* plane. Now we are able to draw a perpendicular *kl* (Fig. 235): the line segment *kl* represents the required distance from the point *K* to the line *AB*.

Figure 236 shows the solution of the same problem by revolving the system consisting of a point *K* and a straight line *AB* first about an axis perpendicular to the *H* plane, and then about one perpendicular to the *V* plane. (The axes are not shown in the drawing—see Sec. 36). Since during the first revolution the horizontal projection of the system changes only its position, but not the configuration and size, then drawing the perpendicular kl, we construct the horizontal projection a_1b_1 in the required position. Using this projection, we find the vertical projection $a_1'b_1'2_1'k_1'$. During the second revolution we must preserve the configuration and size of this projection. We 'attach' the point k_1' to $a_1'b_1'$ by means of a perpendicular $k_1'2_1'$ and construct the projection $a_2'b_2'2_2'k_2'$. Using this projection, we obtain the projection k_2 of the point K and the point designated by two letters (a_2 and b_2) which is the projection of the line segment AB. The required distance from the point K to the line AB is represented by the segment k_2a_2 (k_2b_2).

6. Determine the distance from a point to a plane.

Figure 201 shows the solution for the case of a horizontal projecting plane. The solution is reduced to drawing the perpendicular ak.

Figure 237 gives the solution for an oblique plane specified by a triangle (left), and by the traces (right). Here the method of replacing projection planes is applied: we introduce an additional plane S perpendicular both to the H plane and to the given plane which, finally, turns out to be perpendicular to the plane S (see Figs. 205 and 206 and the corresponding explanations). The required distance is determined by the perpendicular drawn from the point k_s to the projection b_sc_s (Fig. 237, left) and to the trace P_s (Fig. 237, right).

7. Determine the distance between two parallel planes.

The solution of this problem can be reduced to determining the distance from a point taken in one of the planes to the other plane or to introducing into the system V, H an additional plane of projections perpendicular to the given planes, as it is done in Fig. 237.

8. Determine the distance between two parallel lines.

The solution of this problem may be reduced to finding the distance from a point taken on one of the lines to the other line (see Figs. 235 and 236).

Figure 238 shows a construction in which a plane defined by parallel lines is revolved about one of its horizontal (or vertical) lines to a position in which the plane, and consequently, the given lines, are parallel to a projection plane.

The revolution is carried out about a horizontal line KM. It is sufficient to find the new position of, say, the point A (point a_1 on the horizontal plane): the line a_1k and the line parallel to it and drawn through the point m represent the horizontal projections of the given parallel lines when the plane defined by them is arranged parallel to H.

Figure 239 shows how the same problem is solved by the method of replacing projection planes. First both lines are projected on the plane S which is parallel to them (the plane S is passed through one of the lines, AB). Then the lines are projected on the plane T which is perpendicular to them. On this plane the projections of the lines are represented by points. The line segment a_tc_t (or b_td_t) determines the required distance between the given lines.

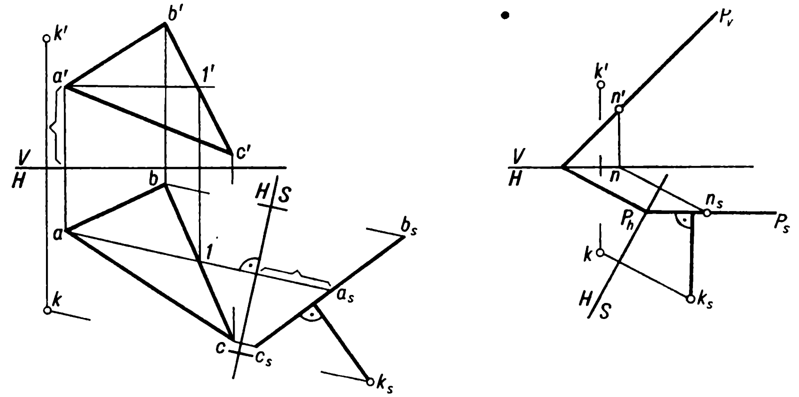

Fig. 237

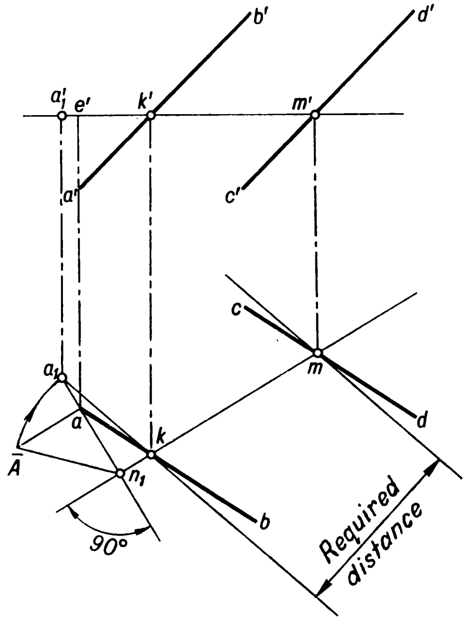

Fig. 238

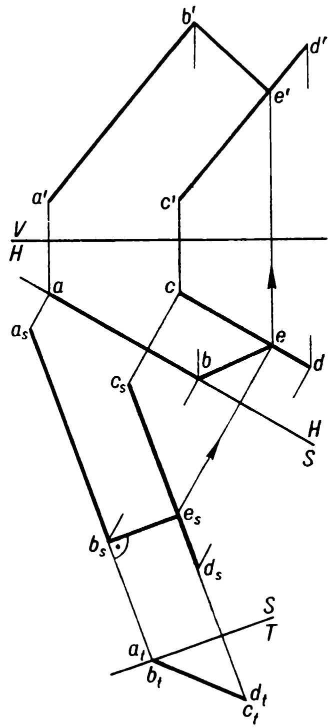

Fig. 239

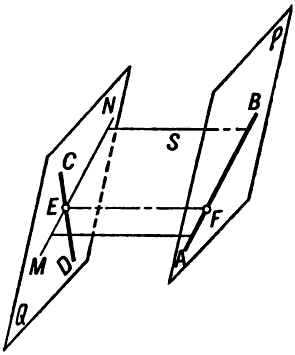

Fig. 240

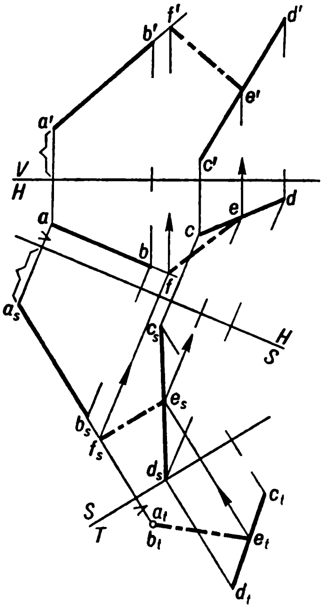

Fig. 241

Shown in the same figure are the projections of the line segment determining the distance between the given lines. The projection on the plane S is drawn through the point b_s (some other point on a_sb_s might be taken as well) parallel to the axis S/T, since in the system S, T the projection on the plane T represents the true length of the distance between AB and CD. The further reasoning is clear from the drawing. The projection on the plane T must be longer than each of the projections b_se_s, be, and $b'e'$.

9. Determine the shortest distance between the skew lines and draw the projections of a perpendicular common to them.

We would remind the reader that the shortest distance between two skew lines is at the same time the distance between the parallel planes containing these lines.

Figure 240 shows a common perpendicular to the skew lines AB and CD.

If parallel planes P and Q are passed through AB and CD, and then through one of them, say through AB, a plane S perpendicular to P and Q is passed, and the line of intersection of the planes S and Q is found (this line MN is parallel to AB), then the required perpendicular to the lines AB and CD will pass through the point E of intersection of the lines CD and MN.

In the drawing shown in Fig. 241 one of the skew lines (AB) is projected into a point on an additional plane of projection (T). The constructions are to be carried out in the following succession:

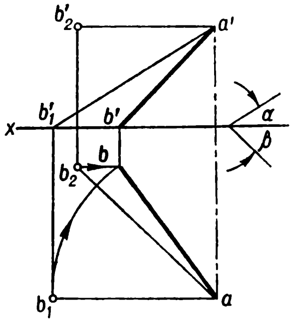

Fig. 242

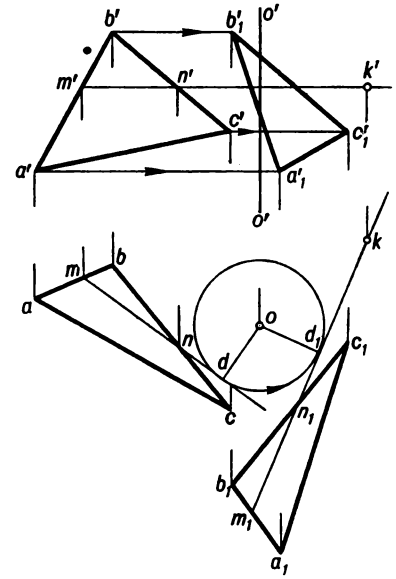

Fig. 243

(a) From the system V, H pass over to a new system S, H where S is perpendicular to H and parallel to AB.

(b) From the system S, H pass over to a new system S, T in which T is perpendicular to S and AB.

(c) Obtaining on the plane T the projection of the line AB in the form of a point and the projection of the second line ($c_t d_t$) and drawing a perpendicular from a_t (b_t) to $c_t d_t$, find the required distance between the given skew lines AB and CD.

Furthermore, the figure shows how the projections of a common perpendicular to AB and CD are constructed. The projection $e_s f_s$ is drawn parallel to the axis S/T.

10. Construct the projections of a segment of an oblique line forming an angle α with the H plane and an angle β with the V plane. Such construction was shown in Sec. 13 (Figs. 73 and 74) but without using the methods set forth in Chapter 5. Now we are going to solve this problem applying the method of revolution.

Suppose (Fig. 242) the line must pass through the point A at an angle α to the H plane and at an angle β to the V plane. As is known (see Sec. 13), for an oblique line the sum of α and β must be less than 90°.

Through the point A two straight lines are drawn: one parallel to the V plane at an angle α to H, the other parallel to the H plane at an angle β to V. We then lay off equal line segments $a'b_1' = ab_2$ on both lines. Let us now revolve the segment AB_1 about an axis perpendicular to H, and the segment

AB_2 about an axis perpendicular to V, both these axes passing through the point A (thus enabling us to keep this point in its given position). At a certain moment the segments will coincide yielding the segment AB (see Fig. 242) which represents the required line. Four such lines can be drawn through the point A.

11. Construct an oblique plane passing through a point A at given angles to H and V.

In Section 29 we established the relationship between the angles formed by an oblique plane with the projection planes H and V, and the angles formed by a perpendicular to this plane with the same projection planes. According to these relationships, to construct a plane at an angle α_1 to H and β_1 to V, we first have to construct a straight line at an angle $\alpha = 90°-\alpha_1$ to H and at an angle $\beta = 90°-\beta_1$ to V (see Problem 10), and then through the given point A to pass a plane perpendicular to the constructed line*.

12. Revolve an oblique plane specified by the triangle ABC (Fig. 243) about a given vertical axis so that this plane passes through a given point K. If the plane passes through the point K, then the latter will turn out to be contained in the plane on one of its horizontal lines. We can at once indicate the horizontal line which, on revolving the plane, must pass through the point K: to this end it is sufficient to draw the vertical projection of the horizontal line through the point k'. On constructing the horizontal projection of the horizontal line (mn) and determining the radius of revolution (od), we draw a circle with respect to which the horizontal projection of the horizontal line will be tangent at any position when the plane is revolved about the given axis. If now a line is drawn from the point k tangent to this circle (kd_1), we may take it for the horizontal projection of the horizontal line, on which the point K must be located when the plane will pass through it.

Having constructed the horizontal projection of the horizontal line after the revolution (m_1n_1), we construct the horizontal projection of the triangle which changes only its position but remains unchanged by configuration and size ($a_1b_1c_1$ is congruent to abc). Using the projection $a_1b_1c_1$, we find the projection $a_1'b_1'c_1'$.

Here we confine ourselves to one solution. A second solution is obtained by drawing a second tangent line from the point k.

The present problem may be modified in the following way: revolve an oblique plane about a vertical axis so that the given point turns out to be contained in this plane.

This problem differs from the preceding one only by that we ourselves have to choose the axis of revolution. May it be chosen arbitrarily?

It turns out that *not any* of the straight lines perpendicular to the H plane may be taken as an axis suitable for solving this problem.

It follows from Fig. 243 that the horizontal projection of the axis of revolution must be arranged so that relative to the horizontal projections of

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*Obviously, on constructing the line in the way considering in Sec. 13, we may pass to it a perpendicular plane which will be the required one.

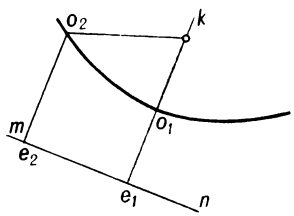

Fig. 244

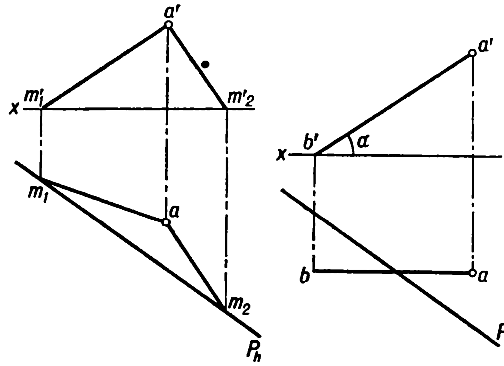

Fig. 245

the point K and the horizontal line MN the circle with the centre o touching the line mn does not contain inside itself the point k, since from the latter point we have to draw a tangent line to this circle.

Hence, at all events the distance of the required point o from the point k must be not less than the distance of the same point o from the line mn. If we take a point o such that the distances are equal to each other (for instance, at points o_1 or o_2 in Fig. 244), then it is possible to place the axis of revolution at such a point.

Where in the drawing will lie all the points equidistant from the point k and from the line mn? As is known, they are located on a *parabola* whose focus is at the point k and mn serves as its directrix. The points situated inside this parabola are nearer to the focus than to the directrix and are unfit for the horizontal projection of the axis of revolution; the points lying on the curve itself, or outside it may be chosen for this purpose.

13. Through a point contained in some plane draw in this plane a straight line at a given angle α to the H plane.

Suppose the plane (designated as P) is specified by two intersecting lines (Fig. 245, left) and we have to draw the required line through the point A at which the lines intersect.

First we find the horizontal trace of the plane P. To this end we draw the x-axis and find the horizontal traces of both lines defining the plane P. The trace P_h passes through the determined traces. If the required line AB was parallel to the V plane, then the angle between the projection $a'b'$ and the axis of projection would be equal to the angle between the line and the H plane. Therefore we have to draw through a' (Fig. 245, right) a straight line at a given angle α to the axis of projection.

The point b' may be taken on this line arbitrarily; for the sake of simplicity it is taken on the x-axis. Then we construct the horizontal projection ab corresponding to the obtained line segment $a'b'$. The projection ab must be parallel to the axis of projection, since the line is positioned parallel to the V plane.

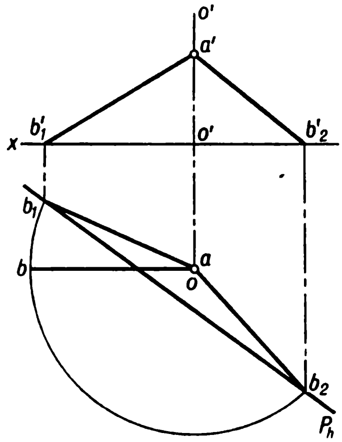

Fig. 246

The line ($a'b'$, ab) thus constructed satisfies one condition, namely, it is drawn at a given angle α to the H plane, but it dissatisfies the other condition: it is not contained in the given plane. To place the line AB in the plane P, preserving at the same time the angle α, it is necessary to resolve it about an axis perpendicular to the H plane. Since the point A lies in the plane P, we have to take the axis of revolution passing through the point A (Fig. 246). During this revolution the point B will move in the plane H, and at the moment AB enters the plane P the point B will be on P_h of this plane. Therefore, revolving the line ab about the point o (a) we 'bring' the point b on the trace P_h and, using the new found position of the horizontal projection, we determine the new position of the projection on the V plane.

As is seen from Fig. 246, the problem has two answers, and its solution is possible if the given angle α does not exceed the angle of inclination of the plane P itself to the H plane. If these angles are congruent, then we obtain only one answer.

14. Find the true size of a plane angle.

The solution of this problem can be seen in Figs. 203 and 210 where the constructions are accomplished by the method of replacing projection planes (the triangle is projected on the additional plane which is parallel to it and its angles are thus determined). In Figs. 223 and 227 the true size of a plane angle is determined by the method of revolution, while in Figs. 230 and 234 the true size of an angle between the traces of a plane in the first quadrant is found by bringing the plane into coincidence with the corresponding plane of projection.

15. Bisect a plane angle. The construction of the bisector of an angle was dealt with in Sec. 15 where we considered the cases of specifying an angle where the constructing of the bisector of an angle of a projection correspond-

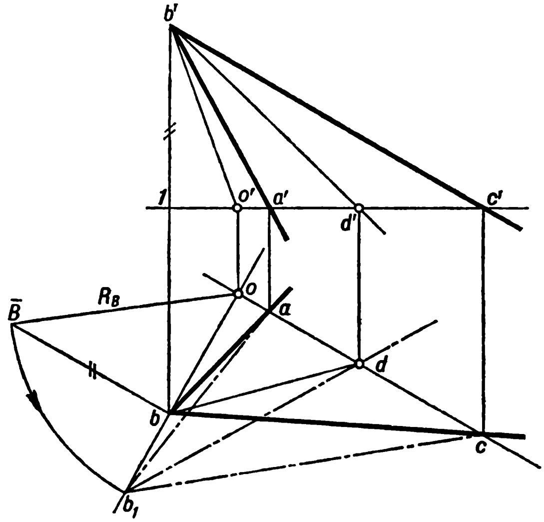

Fig. 247

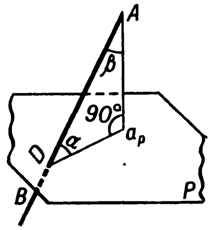

Fig. 248

ed to bisecting the angle in space. Here the general case is considered. The solution is given in Fig. 247.

The plane defined by the sides of the given angle should be positioned parallel to one of the projection planes. Then the angle will be projected on this plane without distortion and can be bisected. In Figure 247 the plane containing the angle is revolved to a position parallel to the *H* plane with the aid of a horizontal line *AC* which is constructed in the drawing. The revolution of the triangle *ABC* about the horizontal line *AC* is reduced to revolving one of its vertices, namely, the point *B*. The centre of revolution is obtained at the point *O* (the projections o' and o); the true length of the radius of revolution R_B is determined as a result of constructing a right-angled triangle $ob\bar{B}$ in which the leg *ob* represents the horizontal projection of the radius of revolution, and the leg $b\bar{B}$ is equal to the line segment $b'1$.

The point b_1 is then joined to the points *a* and *c* which are the horizontal projections of the points located on the axis of revolution and belonging to the sides of the angle. The new horizontal projection, i.e. the angle ab_1c equal to the given angle *ABC* is now bisected to obtain the point *d* on the horizontal projection of the horizontal line and then its corresponding projection d' on the line $a'c'$. The points *d* and d' represent the projections of a point situated on the axis of revolution *AC* and which is thus a 'fixed' one. The lines $b'd'$ and *bd* are the projections of the required bisector.

16. Find the true size of the angle between a straight line and a plane.

Figures 202 and 219 show how the true size of an angle between an oblique line and a plane of projection is determined. Let us now consider the solution for the case of an oblique plane.

If it is required to determine only the magnitude of an angle between a straight line and a plane, then there is no need to construct the projections of this angle*.

Indeed, the magnitude of the angle between the line AB and the plane P (Fig. 248) can be determined by constructing the angle β and finding its magnitude: the required angle $\alpha = 90°-\beta$. In this event the solution of the problem is considerably simplified, since all the constructions aimed at finding the points D and a_p are avoided.

The construction is given in Fig. 249. Drawing from the point A of the line AB a perpendicular to the plane P, we construct the projections of the angle which is a complementary one to the required angle between the line AB and the plane P. We then draw the horizontal line CB and, revolving the plane defined by the angle CAB about it, bring this plane to a position in which it is parallel to the H plane. The new horizontal projection of the angle ca_1b is congruent to the angle CAB. Finally, we have to construct the angle complementary to the angle ca_1b (the angle α in Fig. 249). It is equal to the required angle between the line AB and the plane.

If a plane is specified not by traces, but, for instance, by a triangle, then to draw a perpendicular to it we have to construct a horizontal or a vertical line in the triangle (see Sec. 29).

17. Determine the true size of an angle between two planes.

Figure 250 shows the solution without constructing the projections of the plane angle which measures the dihedral angle formed by the planes P and Q**. This way of solution is especially convenient when the planes are specified by their traces.

If perpendiculars are drawn from some point to the faces of the dihedral angle, then the required plane angle will be equal to the difference between the angle of 180° and the angle formed by these perpendiculars. To determine the angle between the planes P and Q the following constructions are carried out in Fig. 250:

(a) from a certain point K there drawn two perpendiculars: one to the plane P, the other to the plane Q;

(b) revolving about a horizontal line, the angle formed by the perpendiculars is arranged parallel to the H plane.

The required angle between the planes P and Q is equal to the found angle β or (if β is obtuse) to the difference between 180° and the found angle.

Figure 251 represents the solution obtained by applying the method of replacing projection planes. Here we determine the magnitude of the dihedral angle formed by the triangular faces ABC and ABD, AB serving as the edge. If AB turns out to be perpendicular to an additional plane of projection, then both faces will be projected on this plane into line segments, the angle

*For constructing the projections of an angle between a straight line and a plane see Sec. 31.

**For constructing the projections of the plane angle of a dihedral angle see Sec. 31.

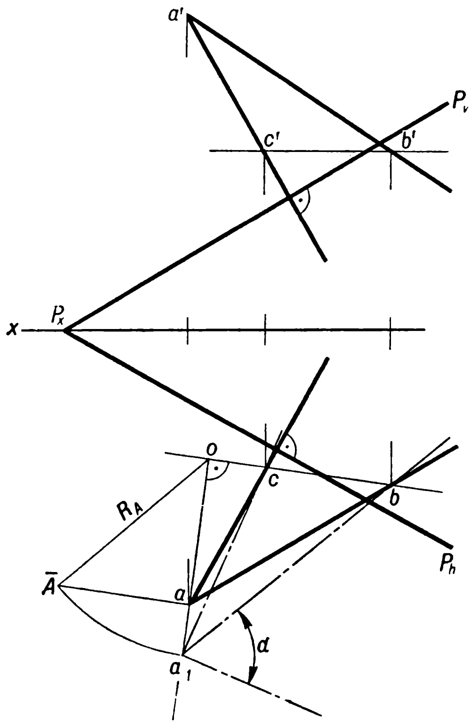

Fig. 249

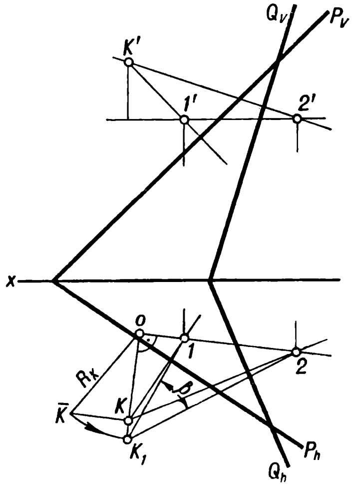

Fig. 250

between which is equal to the plane angle of the given dihedral angle (Fig. 252).

The construction in Fig. 251 is accomplished according to the following scheme: from the system V, H to the system S, H in which S is perpendicular to H and parallel to AB, and then to the system S, T where T is perpendicular to S and AB. Shown on the plane S are only the projections of the points A, B, C, and D; the faces ABC and ABD are not outlined.

Determining the true size of the angles between an oblique plane and the projection planes H and V by the method of replacing planes of projection was shown in Figs. 205, 206, and 207. Application of the method of revolution is illustrated in Fig. 221 (the angle with the V plane).

18. Determine the true size of a triangles.

The solution by the method of replacing projection planes can be found in Figs. 203 and 210, the same by the method of revolution in Figs. 223 and 227.

19. Revolve a point A about the axis MN through an angle α clockwise when viewed from M to N (Fig. 253).

The construction is carried out by the method of replacing projection planes. The plane in which the point A is revolved and the plane of projection

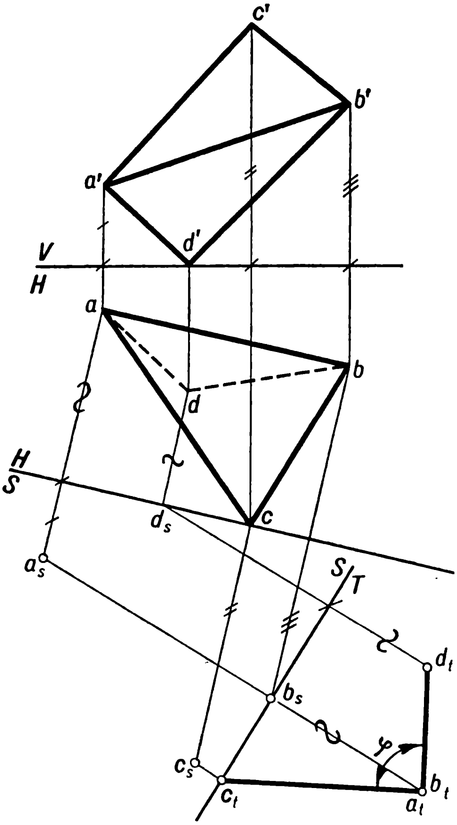

Fig. 251

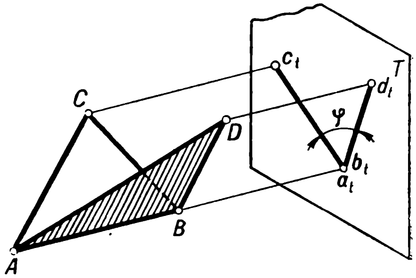

Fig. 252

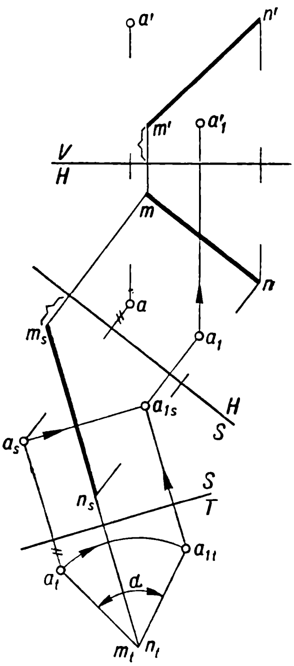

Fig. 253

T are brought into mutually parallel positions by successively forming new systems of projection planes in the following succession: from the system *V*, *H* to the system *S*, *H* where *S* is perpendicular to *H* and parallel to *MN*, and finally to the system *S*, *T* in which *T* is perpendicular to *S* and *MN*. Accordingly, the revolution of the point *A* is represented as the revolution of the projection a_t about the centre m_t (n_t) through a given angle clockwise (since, by hypothesis, to determine the direction of revolution we must view from the point *M* towards the point *N*). We then obtain the projection a_{1s} on the line drawn through a_s perpendicular to $m_s n_s$, and then the projections

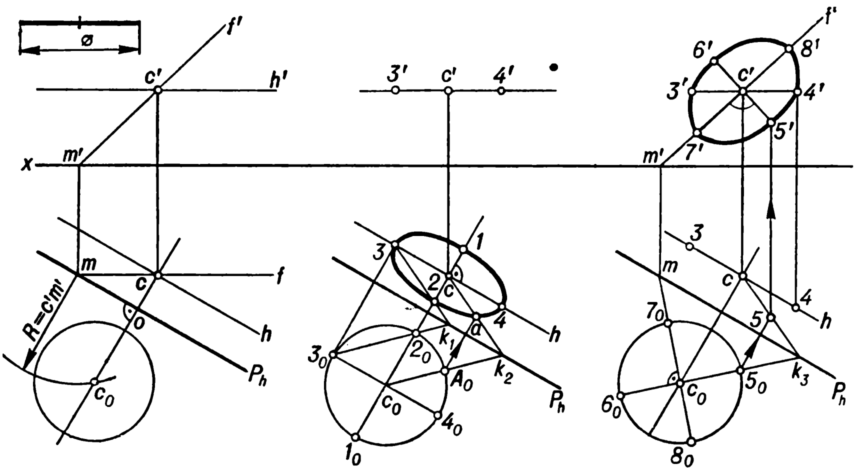

Fig. 254

a_1 and a_1' which corresponds to the displacement of the point A to the position A_1.

20. Construct the projections of a circle of a given diameter contained in an oblique plane.

The solution is given in Fig. 254, where the method of revolution is applied.

Let the plane (designated P) containing the circle be specified by a horizontal line with the projections $c'h'$ and ch, and a vertical line with the projections $c'f'$ and cf. The centre of the circle is located at the point C.

In the first position (Fig. 254, left), taking the x-axis as given, and finding the horizontal trace of the vertical line CF (point M), we draw the trace P_h parallel to the projection ch of the horizontal line. We then find the coincident position of the centre C (point C_0) on the H plane and construct in this plane a circle of the given radius with this point as centre.

The required projections of the circle are ellipses. Figure 254 shows how the axes of these ellipses are constructed for each projection of the circle.

The major axis of the ellipse which is the horizontal projection of the circle lies on the horizontal projection of the horizontal line, and $c3 = c4 =$ $=$ the radius of the circle (see Fig. 254, middle). The minor axis is obtained with the aid of the diameter $3_0 4_0$ which is parallel to the trace P_h and of the diameter $1_0 2_0$ perpendicular to this trace. Point 2 is obtained with the aid of the line $3_0 k_1$, while point 1 on the same projection can be constructed on the basis of that $c2 = c1$.

The major axis $7'8'$ of the vertical projection of the circle (Fig. 254, right) is found on the vertical projection of the vertical line. Line segments $c'7'$ and $c'8'$ equal to the radius of the circle are laid off from the point c'. To the axis $7'8'$ there corresponds the diameter $7_0 8_0$ of the circle located on the vertical line MF brought into coincidence with the H plane.

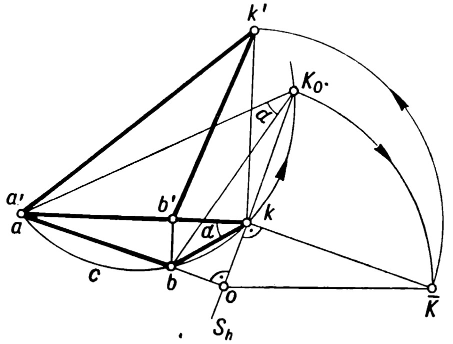

Fig. 255

The minor axis $5'6'$ on the vertical projection is drawn perpendicular to $7'8'$. Point $5'$ is constructed with the aid of point 5_0 of the diameter 5_06_0 of the circle drawn perpendicular to the diameter 7_08_0 extended to intersect the trace P_h at point k_3. On the horizontal projection of the auxiliary line ck_3 we find the projection 5 and then construct $5'$. Laying off a line segment $c'6'$ equal to the segment $c'5'$, we get the projection of the minor axis $(5'6')$.

With the axes thus obtained for both ellipses, we may pass over to constructing the ellipses themselves by its points. We can get these points in the way shown in Fig. 254 (middle) for the point A, the projections a and a' being constructed analogously to the points 5 and $5'$.

21. Construct the vertical projection of an angle whose true size is equal to its horizontal projection.

In Section 15 it was established that the projections of an acute (or obtuse) angle contained in an oblique plane may be congruent to the angle projected.

Suppose that the angle akb is the horizontal projection of some angle α (Fig. 255). We draw the horizontal trace of the plane containing the angle under consideration (the line ab) and revolve the point K about it until it is coincident with the plane H. If a circle is drawn through the points a, b, and k, then any angle (including the angle aK_0b) inscribed in this circle and subtended by the arc acb will be equal to α. Obviously, point K_0 is the point K (the vertex of the angle AKB) brought into coincidence with the H plane. It is obtained as the point of intersection of the arc drawn through the points a, b, and k with the trace S_h of the plane of revolution of the point K, the line segment oK_0 being its radius of revolution. Erecting a perpendicular to ok at point k, and intersecting this perpendicular with an arc of radius oK_0, we get the point $\bar{K}$ and line segment $k\bar{K}$ representing the distance of the point K from the H plane, i.e. the distance of the projection k' from the x-axis. The angle $a'k'b'$ represents the vertical projection of the angle AKB which is equal to its horizontal projection akb.

This and some previous sections have been devoted to the solution of problems involving the construction of the common elements of the various geometric figures (for instance, the construction of the point of intersection of a straight line and a plane, or Problem 1 of this section). Such problems are called 'positional' as opposed to metrical problems in which the length of line segments, angles, areas of surfaces, etc. are determined.

QUESTIONS TO SEC. 38

1. In what succession are the axes of revolution to be taken to bring an oblique line to the position in which it will be arranged perpendicular to the *H* plane? to the *V* plane?
2. How are the true length of a segment of an oblique line and its angles with the planes *H* and *V* determined?
3. How is the distance from a point to an oblique line determined?
4. How is the distance from a point to an oblique plane (to a profile plane) determined?
5. How is the distance between two parallel planes (between two parallel lines; between skew lines) determined?
6. Is it possible to construct the projections of a segment of an oblique line by the method of revolution given the angles of its inclination to the planes *H* and *V*? If it is possible, then how is this done?
7. What does the parabola constructed in Fig. 244 mean?
8. How is the true size of a plane angle found?
9. How is the bisector of an angle constructed in the drawing?
10. How can we find the true size of an angle between a straight line and a plane?
11. How can we find the true size of an angle formed by two planes?
12. How do we construct the projections of a circle contained in an oblique plane?

CHAPTER 6

POLYHEDRONS

Sec. 39. Constructing the Projections of Polyhedrons

The construction of the projections of a polyhedron on a plane is reduced to constructing the projections of points. For instance, when projecting a pyramid *SABC* on the *V* plane (Fig. 256, left), we construct the projections of the vertices *S*, *A*, *B*, and *C*, and as a consequence, the projections of the base *ABC*, faces *SAB*, *SBC*, *SAC*, edges *SA*, *SB*, etc.

Also, when projecting a trihedral angle* with the vertex *S* (Fig. 256, right), we, besides the vertex *S*, take one point on each of the edges of the angle (points *K*, *M*, *N*) and project them on the *V* plane. As a result, we get the projections of the edges and faces (plane angles) of the trihedral angle and, on the whole, the angle itself.

Figure 257 represents a polyhedral solid $ACBB_1D\ldots$ (i.e. a portion of space bounded from all the sides by plane figures—polygons) and its projection on the *H* plane, i.e. the figure $acf_1e_1d_1def$. Any point located inside the outline of this figure is the projection of at least two points on the surface of this solid. For instance, the point designated by two letters *m* and *n* serves as the projection of the points *M* and *N* lying on a common projecting line.

A point lying on the outline of the projection is the projection of one point (for instance, *a* is the projection of point *A*), or several points and sometimes a set of points (for instance, *b* is the projection of not only the point *B* but also of a set of points of the face *ABC* situated on the projecting line *Bb*).

The totality of projecting lines passing through all the points of the outline of the projection form a *projecting surface*. The given solid is enclosed by this surface touching it from inside. For the solid represented in Fig. 257 the projecting surface consists of the planes S_1, S_2, S_3, etc. The line of contact of the projecting surface with the solid is called the contour of the solid

*In this case a convex one, i.e. a dihedral angle which lies entirely on one side of any plane containing one of its faces extended infinitly.

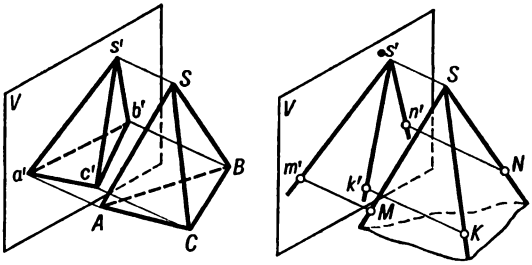

Fig. 256

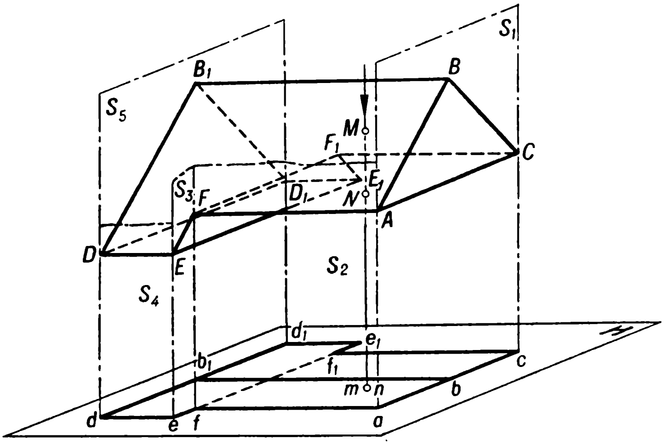

Fig. 257

with respect to the chosen plane of projection. In Fig. 257 the contour is represented by the polygonal line $ACF_1E_1D_1DEFA$*.

In parallel projecting the projecting surface is cylindrical (see Sec. 1). If the contour of a solid contains rectilinear segments with reference to the plane of projection, then the projecting surface for each of such segments is reduced to a plane.

The line segment bb_1 drawn in the figure is the projection of the edge BB_1 visible with respect to the H plane. It is obligatory to show all visible edges on the projection of a solid.

The projection of the line segment FF_1 is obtained inside the outline of the projection. It is drawn with a short-dash line, since, when projected on the H plane, the points of the line segment FF_1 are invisible.

*We may consider that in this case all points of the line segments AB, BC, DB_1, B_1D_1, EF, and E_1F_1, and even the areas of the triangles ABC and DB_1D_1, and portions of the trapezoid EFF_1E_1 belong to the contour of the solid, since the projecting planes S_1, S_3, and S_5 pass through these figures, respectively.

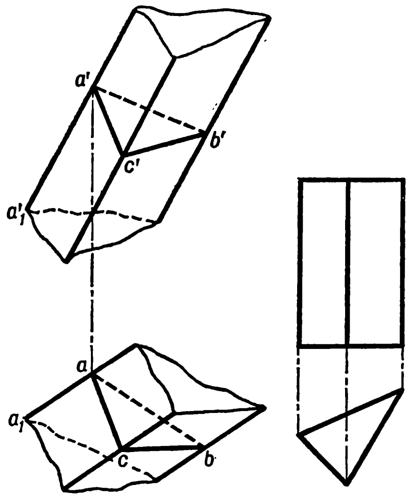

Fig. 258

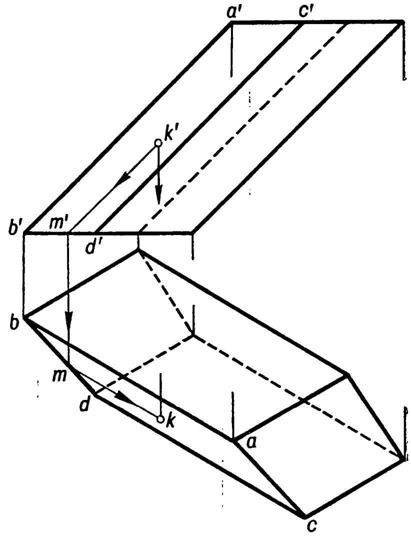

Fig. 259

The construction of the projection of a *faced surface* is also reduced to the construction of the projection of some points and straight lines belonging to this surface. The projection of the surface bounding a solid has an *outline* which coincides with the outline of the projection of this solid. When representing an infinitely extending surface, a certain portion of this surface is usually separated with wavy lines thus establishing a conventional contour with respect to the plane of projection.

Sec. 40. The Drawings of Prisms and Pyramids

Suppose we know the shape and position of the figure obtained as a result of intersecting all of the lateral faces of a prism by a plane, as also the direction of its edges (Fig. 258). This is the way a prismatic surface is usually specified. Intersecting a prismatic surface with two parallel planes, we get the bases of a prism (Fig. 258). A prism may be specified by one of its bases and the altitude or the length of the lateral edge.

When positioning a prism for its representation, it is advisable to arrange its bases parallel to the plane of projection.

What indications enable us to determine that nothing else but a prism (or, in particular, a parallelepiped) is represented in a given drawing? The answer is: the drawing contains only rectilinear segments* which serve as projections of either edges, or faces; parallelograms or rectangles as the

*The condition common to all polygons.

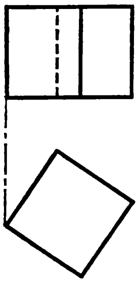

Fig. 260

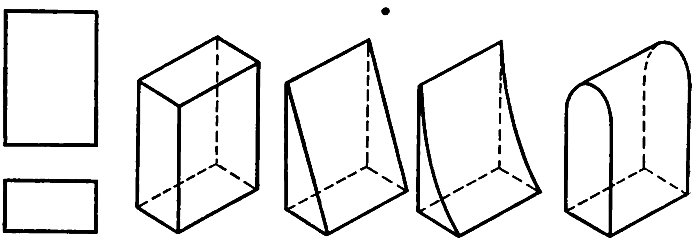

Fig. 261

projections of the lateral faces; and any polygon as the projection of the base.

Examples are given in Figs. 258-60 where represented in the system V, H are a right triangular prism, an oblique quadrangular prism, and a cube (equal edges and rectangular faces testify to the fact that it is a cube).

But it would be a mistake to assert that the solid represented in Fig. 261 is necessarily a prism or a parallelepiped in spite of the fact that some of the above discussed indications are present. Shown in Fig. 261 (right) are possible solutions. Obviously, to define more exactly in this case the profile projection or designation of the vertices should be given.

Figure 262 depicts an irregular quadrangular prism (with trapezoids as its bases). Figure 263 (top) shows the profile projection of this prism constructed with the aid of an auxiliary straight line. The same figure (bottom) represents the prism referred to the coordinate planes coinciding with its faces. In this case the third projection is constructed using the coordinates of the vertices.

To specify the surface of a pyramid we must have the section figure cut off by a plane intersecting all the lateral faces of the pyramid and the point of their intersection. A pyramid is usually specified in the drawing by the projections of its base and vertex, and a frustum of a pyramid by the projections of both bases.

When positioning a pyramid for its representation it is advisable to arrange its base parallel to the plane of projection. Figure 264 demonstrates an irregular triangular pyramid with the base parallel to the H plane represented in the system V, H. The drawing gives a clear representation of the shape of the base and lateral faces. Two projections are sufficient, in general, for a pyramid, provided one of them shows the shape of the base. But considering the solid represented in Fig. 265, it would be a mistake to assert that it is nothing else but a pyramid, since we are misled by both given projections. Here in the system V, H the form of the line located in the profile plane remains undetermined. This line may be a curve and, consequently, the faces containing this curve will not be plane figures (see the right-hand picture in Fig. 265). Obviously, the profile projection here is of great help, it would give a definite answer to the question whether the given solid is a pyramid or not.

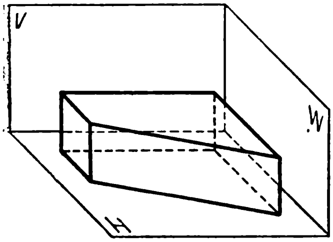

Fig. 262

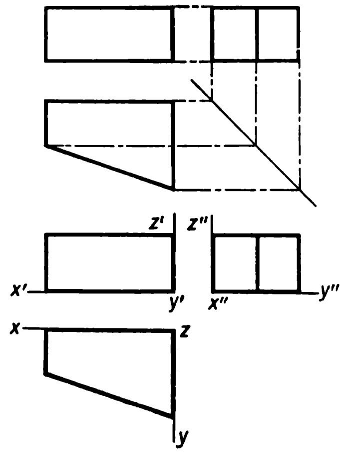

Fig. 263

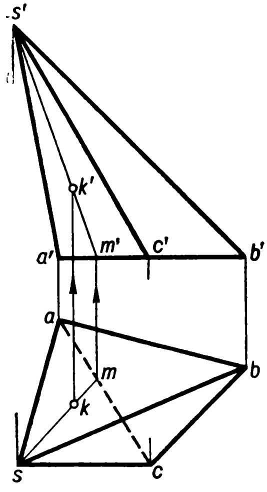

Fig. 264

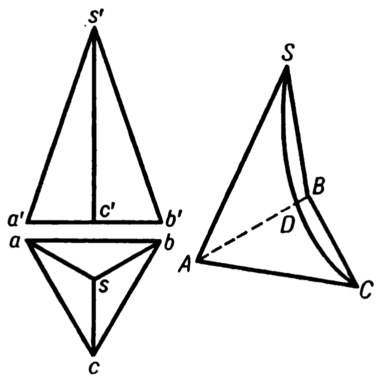

Fig. 265

Figure 266 illustrates how, for instance, the coordinate axes may be taken for a given pyramid: the z-axis is directed along the altitude, the coordinate plane xOy being coincident with the base of the pyramid. The coordinate axes are represented by their projections. With such arrangement of the axes only one coordinate (the z-coordinate) is sufficient to determine the vertex S of the pyramid.

If it is required to construct a point lying on one of the faces of a polyhedron on both of its projections, then this point must be "tied" with the corresponding face by a straight line.

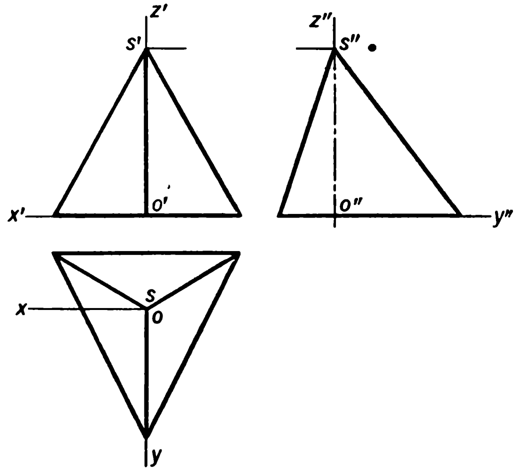

Fig. 266

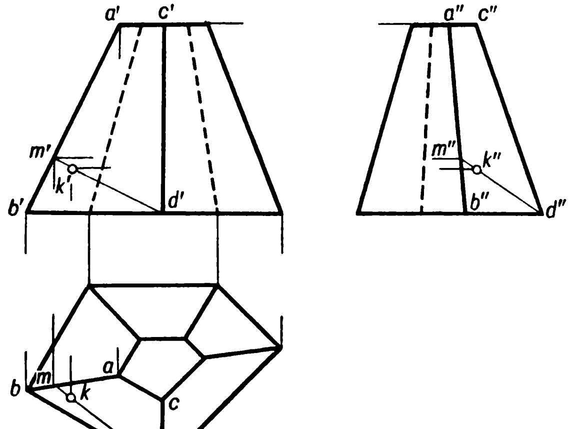

Fig. 267

In Figure 259 point K is constructed on the face $ABCD$ with the aid of a segment of a straight line KM. Suppose it is required to find the horizontal projection of the point K, given its vertical projection k'; moreover, point K must lie on the face $ABCD$. In this case we first construct the vertical projection of the segment of an auxiliary line ($k'm'$), and then the horizontal projection of this segment on which the horizontal projection of the point K is determined. Since the segment $k'm'$ is parallel to $a'b'$, then km is also parallel to ab.

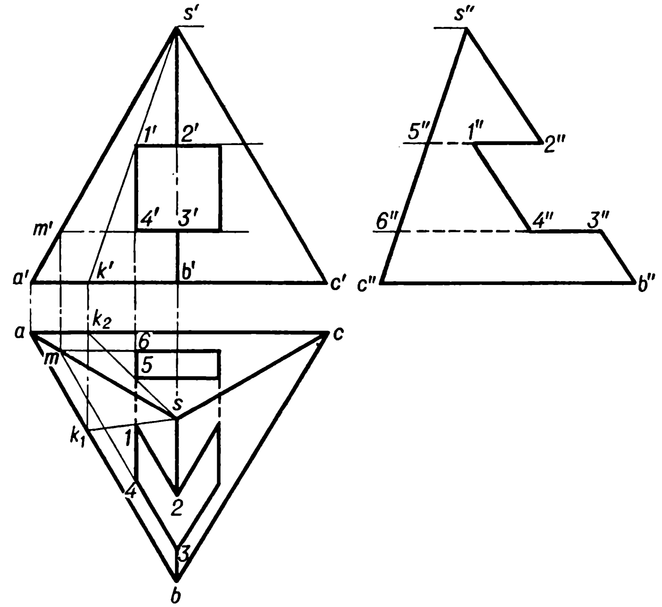

Fig. 268

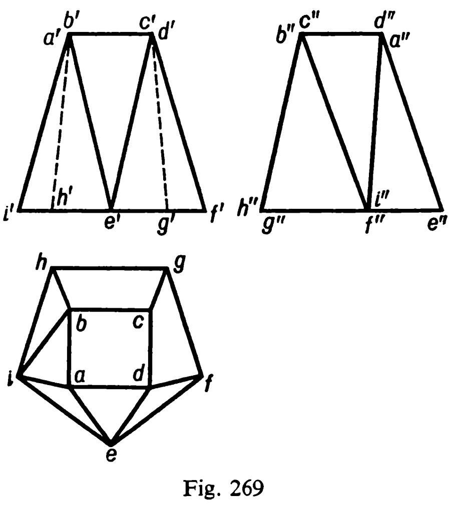

Fig. 269

Figure 264 shows how a point K is constructed on the face SAC by means of a straight line drawn through the vertex of the pyramid. If the horizontal projection k of the point K is given and it is required to find the vertical projection k', then the segment sm has to be first constructed. We then find the point m' with the aid of the point m, obtain the segment $s'm'$ and on it the required projection k'.

Figure 267 represents a frustum of a pentagonal pyramid and shows the construction of a point K on the face $ABDC$ by its projection k given, using a segment of the straight line DM.

In general, an auxiliary line for constructing a point on a face is chosen arbitrarily, but one thing should always be borne in mind: the constructions involved should be as simple as possible.

Figure 268 represents a regular triangular pyramid with a prismatic hole. The construction is carried out using the vertical projection specified completely. We can see in the drawing how points *1* and *5* are constructed on the horizontal projection with the aid of straight lines drawn through the vertex S. Points *3*, *4*, and *6* (on the horizontal projection) are found with the aid of the straight lines drawn on the faces SAB and SAC parallel to the H plane. The horizontal projections of these lines pass through the point m parallel to ab and ac. Point *2* can be found here either in the same way as point *3*, or with the aid of the profile projection.

Figure 269 gives an example of a polyhedron called the prismatoid. The parallel bases of such a polyhedron represent polygons with an arbitrary number of sides, its faces being triangles or trapezoids (in Fig. 269, for instance, a triangle ADE and a trapezoid $BHGC$).

Sec. 41. Systems of Arranging Views in Mechanical Drawings

Mechanical drawings are made according to the principle of *orthogonal projection* which makes it possible to obtain the true shapes and dimensions of the objects to be represented. In other words, the objects are represented in the drawing without distortion.

A combination of properly arranged projections ensures the representation of the shape of an object and its location in space. Each projection presents a picture corresponding to a definite direction of viewer's sight (see Fig. 270).

Technical drawings use various representations conveying different information. They are subdivided into *views*, *sectional views*, and *sections*. Here we shall dwell only on view.

The *view* is defined as a *representation of the visible part of the surface of an object facing the viewer*. Hence, a view represents not the whole object, not all of its faces, edges, etc., but only those facing the viewer. On the contrary, any projection gives a complete representation of an object. Consequently, there is a difference between a projection and a view: the projection represents the whole surface of an object, whereas the view depicts

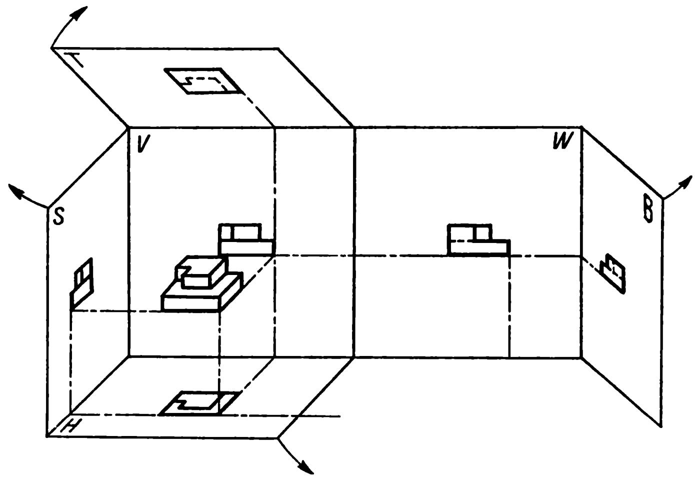

Fig. 270

its portion, i.e. the part facing the viewer and seen by him. But if we realize the indication of the standard *allowing the view to show* (*with short-dash lines*) *the invisible portions of surfaces*, then the difference between a projection and a view exists no longer. For instance, in Figs. 268 and 269 each of the views completely coincides with the corresponding projection.

It was mentioned in Sec. 5 that in making mechanical drawing of various machines and their parts we have to resort to other planes of projection (besides V, H, and W). Figure 271 illustrates six faces of a cube taken for the *main planes of projection*. All of them are brought into coincidence with the plane of the drawing, as is shown in Fig. 270. In space the plane S is parallel to the plane W, T to H, and B to V. With respect to each of the planes the viewer must occupy the same position as he occupies relative to the planes V, H, W, the object must be situated between the viewer and the appropriate plane of projection.

The six *principal views* obtained on the above mentioned planes of projection and arranged as in Fig. 271 are called: *front* or *main view* (on V); *top view* (on H); *left-hand view* (on W); *right-hand view* (on S); *bottom view* (on T); and *rear view* (on B). All these views should always be in horizontal or vertical alignment with the front or top views; projectibility from one view to another is very important.

The obtained arrangement of views corresponds to the *first-quadrant system* (the first dihedron system), or *European system*. This system is used in the USSR and almost in all European countries. There exists another system—the third quadrant system (the third dihedron system)—called the American system which is practical in the USA, Great Britain, the Netherlands, Canada, and some other countries. In this system the plane of projection is supposed to be arranged between the viewer and an object. In

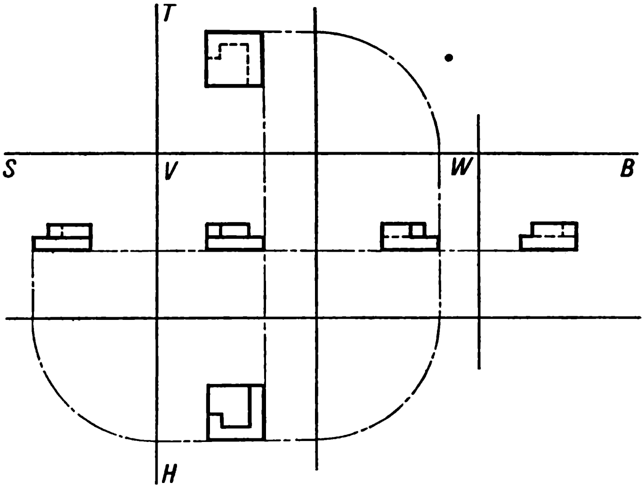

Fig. 271

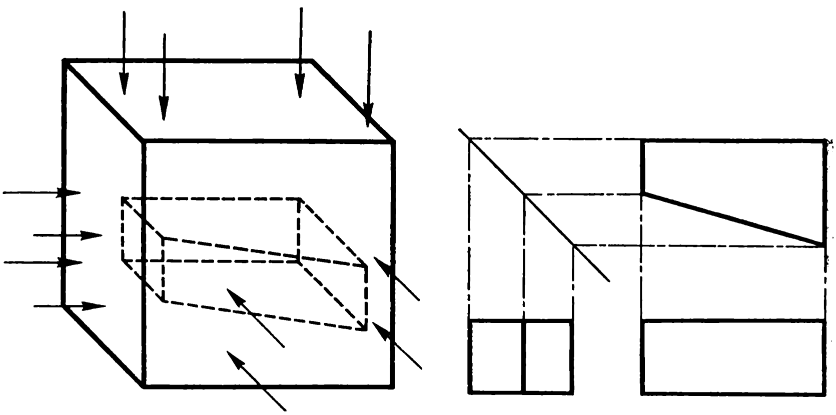

Fig. 272

Figure 272 (left) the prism is situated behind the vertical plane of projection and below the horizontal plane; the profile plane of projection is also shown (i.e. the prism is located in the seventh octant). Arrows indicate the directions of viewer's sight. The object is viewed here as if through "glass" planes. The arrangement of views thus obtained (in this case the *front view*, *top view*, and *left-hand view*) is shown in Fig. 272 (right). Here, as in Fig. 271, the drawing is based on the front view (the *main view*), but the *top view* turns out to be located above the *main view*, and the *left-hand view* is not on the right (see Fig. 268), but on the left of the *main view*.

Thus, regardless of the system used, when drawing the main view, the latter should be placed with respect to the plane V so that it occupies (if possible) the working position and reveals its shape and dimensions to the best advantage. The number of views in an orthographic drawing must be minimum but sufficient to form a complete conception of the shape and dimensions of the object.

Sec. 42. Prisms and Pyramids Cut by a Plane or by a Straight Line

The plane sections of prisms and pyramids are closed figures, the vertices and sides of which are determined by intersections of a given plane with the faces and edges of the given geometrical solid. Hence, to construct sections, one has to find the points of intersection of the edges with a given plane or to construct line segments along which the plane intersects the faces of a prism or a pyramid. In the first case the construction is reduced to the problem of intersection of a straight line and a plane (so-called edge method), in the second case to the problem of mutual intersection of planes (the face method).

When a secant plane is parallel to none of the projection planes, a figure obtained in section is projected in a distorted form. Therefore, if it is required to determine the true size of the section figure, one of the methods considered in Chapter 5 should be applied.

Figure 273 shows the intersection of a right quadrangular prism by a plane specified by two intersecting lines EF and EG. Let us designate this plane by the letter P.

In this case the section figure is a quadrilateral whose vertices represent the points of intersection of the edges of the prism with the plane P. Since the prism is a right one and its base is parallel to the H plane, the horizontal projection of the section is determined at once, without any special construction—it coincides with the projection $abcd$. Obviously, it is possible to find points K and L at which the edges of the prism passing through the points A and D cut out the plane P. It is done with the aid of a plane S containing a face of the prism $S \times P =$ 1-2, whence we obtain points k' and l'. Passing a plane T, we obtain $T \times P =$ 3-4 and points m and n.

Thus, the method of construction illustrated in Fig. 273 is reduced to the use of auxiliary planes S and T passing through the appropriate faces of a prism, and to the construction of line segments KL and MN along which the faces intersect with the plane P.

The vertical projection of the line of intersection consists of a visible and an invisible portions, the visible portion being located on the visible faces facing the viewer.

In Figure 273 the lower portion of the prism situated below the plane P is represented as invisible. The line of intersection is drawn only on the faces of the prism.

If the secant plane is perpendicular to one of the projection planes (as in Fig. 274, left), then the section figure is obtained without any additional

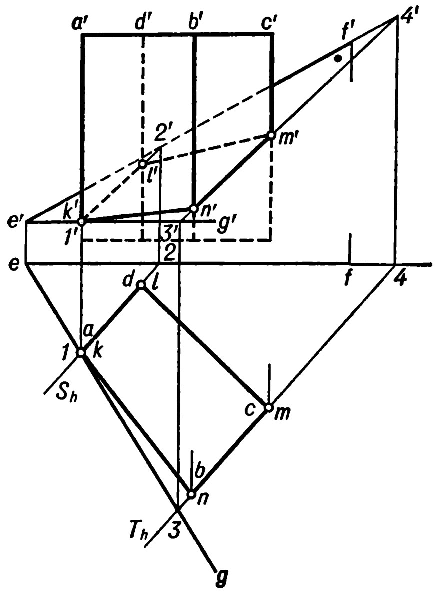

Fig. 273

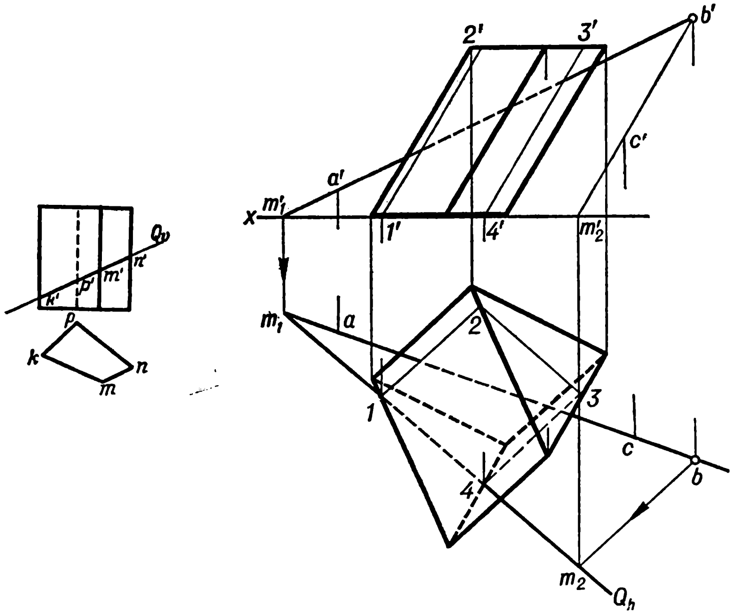

Fig. 274

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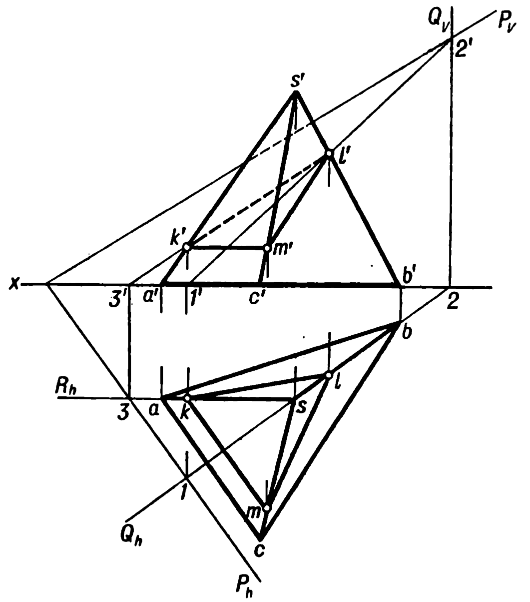

Fig. 275

constructions: the vertical projection $k'p'm'n'$ is situated on the trace Q_v, the horizontal projection ($kpnm$) coinciding with the projection of the prism.

Figure 274 (right) shows the intersection of a prism by a plane Q specified by two intersecting lines AB and BM_2, the latter being parallel to the edges of the prism. Hence, in this case the cutting plane is an oblique one parallel to the edges of the prism. The figure obtained in the section is a parallelogram ***1-2-3-4***, whose sides ***1-2*** and ***3-4*** are parallel to the edges of the prism. To draw these sides, we have to construct the trace of the plane Q on the plane containing the base of the prism and to intersect the base by this trace along the line segment ***1-4***.

Illustrated in Fig. 275 is a pyramid cut by an oblique plane P represented by its traces. The problem is reduced to finding the points of intersection of the edges SA, SB, and SC with the plane P, i.e. to the problem of intersection of a straight line and a plane (see Sec. 25). Let us find the point L at which the edge SB cuts the plane P. Proceed as follows: (1) draw an auxiliary plane through SB, in this case it is a horizontal projecting plane Q; (2) find the line of intersection ***1-2*** of the planes P and Q; (3) find the point L at the intersection of the lines SB and ***1-2***.

Further, since in this case the edge SA is parallel to the plane V, we pass through it an auxiliary vertical plane R. It intersects the plane P along its vertical line with the initial point *3*. The intersection of this vertical line with the edge SA yields a point K.

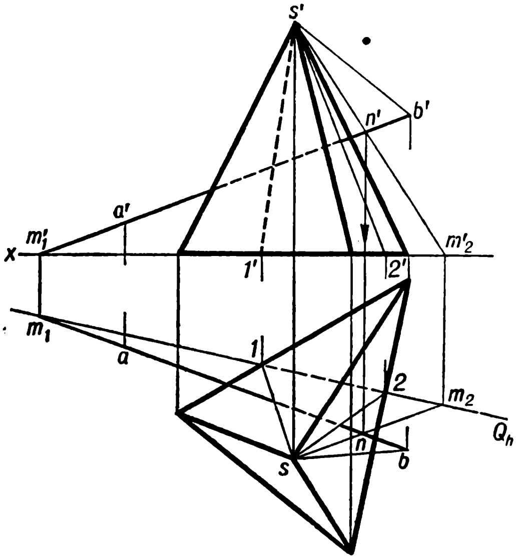

Fig. 276

Let us now draw attention to another peculiarity of this example: the projection *ac* is parallel to the trace P_h. This is the case when two planes have parallel horizontal traces (P_h is parallel to *ac*, but *ac* is a part of the horizontal trace of the plane containing the face *SAC*), and the line of intersection of such planes is their common horizontal line. Therefore we may draw through the found point *K* a straight line parallel to the edge *AC* (or parallel to P_h), and in this way find the point *M*.

If there were no such peculiarities we would proceed as in constructing the point *L*.

In making the drawing of Fig. 275 we proceeded from the assumption that the plane *P* is transparent, and that the main task was to construct the lines on the faces separating the pyramid into two parts.

Let a pyramid (Fig. 276) be cut by a plane *P* specified by two intersecting lines *AB* and *SB*, the latter passing through the vertex of the pyramid. Consequently, the plane *P* cuts off a triangle one of whose vertices is located at the point *S*. To find two other vertices of the triangle (points *1* and *2*), the trace of the plane *P* should be constructed on the plane containing the base of the pyramid. The remaining constructions are obvious from the drawing.

When the surface of a prism or a pyramid is cut by a *straight line* two points are obtained. They are frequently called "*entry*" and "*exit*" *points*. To find these points, we have to pass through the given line an auxiliary plane and to find the lines of its intersection with the faces; these lines on the

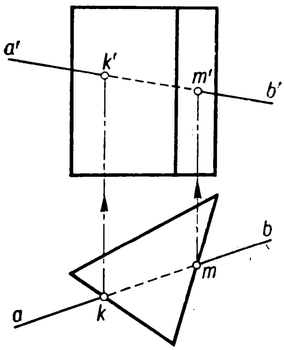

Fig. 277

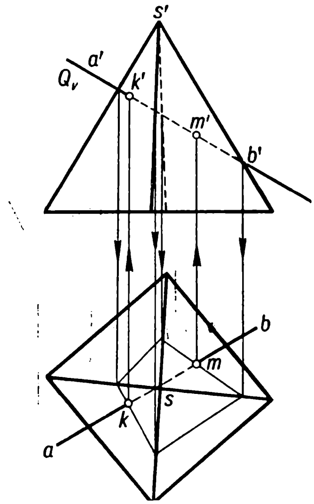

Fig. 278

faces turn out to be contained in one plane with the given line, and their intersections yield points at which the given line cuts the surface.

We may come across cases when there is no need in such constructions. An example is given in Fig. 277 in which the positions of the projections *k* and *m* are obvious, since the lateral faces of the prism are perpendicular to the *H* plane. We find points *k'* and *m'* using the points *k* and *m*.

The construction of the points of intersection of a straight line with the surface of a pyramid is shown in Fig. 278. Through the line *AB* an auxiliary vertical projection plane *Q* is passed. The vertical projection of the figure cut off by this plane coincides with the vertical projection of the plane; the horizontal projection of the section is found by construction. The points of intersection of the horizontal projection of the line *AB* and the horizontal projection of the figure obtained by section represent the horizontal projections of the required points. The vertical projections of the points of intersection (*k'* and *m'*) are constructed with the aid of the found horizontal projections (points *k* and *m*).

We can also construct the points of intersection of a straight line with the surface of a prism in the following way. This time we are going to use *oblique* projecting instead of orthogonal projecting. Let us project the prism and the line *AB* (Fig. 279) on the *H* plane in the direction parallel to the edges of the given prism. The prism will be projected into a triangle $c_0d_0e_0$ coinciding with the horizontal projection of the lower base of the prism, and the line *AB* into a line a_0b_0 which will intersect the sides of the triangle $c_0d_0e_0$

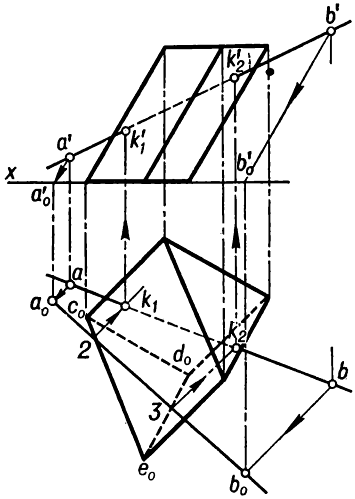

Fig. 279

at points *2* and *3*. Backward projecting yields the projections k_1 and k_2 which help us find k_1' and k_2'.

Thus, we have considered intersections of prisms and pyramids with a plane and a straight line. The relevant constructions are reduced to solving problems on the intersection of two planes, or a straight line and a plane. These problems are of essential importance and are encountered in various cases. They also underlie the construction of lines of mutual intersections of polyhedral surfaces to be considered in the following section.

QUESTIONS TO SECS. 39-42

1. What is the contour of a solid with respect to a projection plane?
2. How is a prismatic surface specified?
3. What indications make it possible to establish that a given drawing represents a prism (or a parallelepiped)?
4. What specifies the surface of a pyramid?
5. What is meant under the term 'tetrahedron'?
6. Under what condition are two projections sufficient to represent a pyramid?
7. What is a prismatoid?
8. What is a *view* in mechanical drawings?
9. What is the difference between a *view* and a *projection* and under what condition is this difference eliminated?
10. What systems of arrangement of views are used in engineering drawings?

11. How do we construct the figure obtained as the intersection of a prism or a pyramid with a cutting plane?

12. How do we construct the points at which a straight line cuts a prism or a pyramid (the points of entry and exit)?

13. Is it possible to establish the generality of the methods of this construction and of the construction of the point of intersection of a plane and a straight line?

14. How is a prism cut by a plane parallel to its lateral edges?

15. How is a pyramid cut by a plane passing through its vertex?

16. How is oblique projecting applied for finding the points of intersection of a prism and a straight line?

Sec. 43. The Mutual Intersection of Polyhedral Surfaces

The line of the mutual intersection of polyhedral surfaces can be constructed by two methods, or by the combination of these methods. Which method is to be applied depends on the initial conditions. It is usually preferable to use the method involving simpler constructions. These methods consist in the following:

(1) *We determine the points at which the edges of one of the surfaces intersect the faces of the other, and the edges of the second surface intersect the faces of the first* (i.e. the problem on finding the intersection of a straight line and a plane). Through the found points a polygonal line is drawn in a certain succession, which represents the line of intersection of the given surfaces. Only the projections of the points lying in one and the same face may be joined by straight lines.

(2) *We determine the line segments along which the faces of one surface intersect the faces of the other* (in other words, the problem on finding the intersection of two planes). These line segments are segments of the polygonal line obtained as the intersection of polyhedral surfaces.

If the projection of an edge of one of the surfaces does not intersect the projection of a face of the other even on one of the projections, then the given edge does not intersect this face. But the intersection of the projection of an edge and a face does not mean yet that the edge and the face intersect in space.

Some of the below examples use the above considered general schemes of constructing the points of intersection, other examples illustrate how particular peculiarities are used for simplifying the constructions involved.

The example given in Fig. 268 may be considered as an intersection of a pyramid and a prism. Points *2* and *3* are obtained as a result of intersection of the upper and lower faces of the prism by an edge of the pyramid, while the lines passing through the points *5* and *6* are yielded by the intersection of the same faces with the face *SAC* of the pyramid.

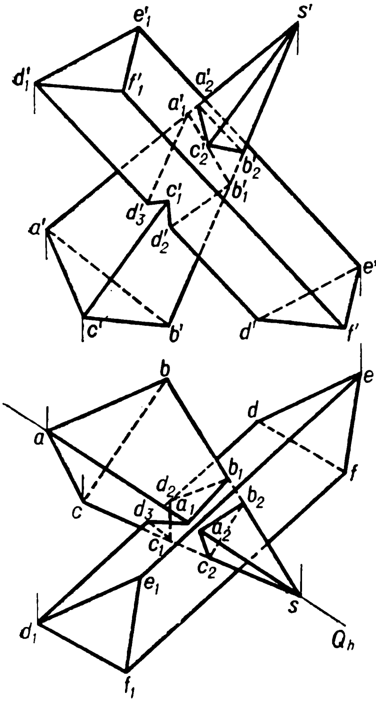

Fig. 280

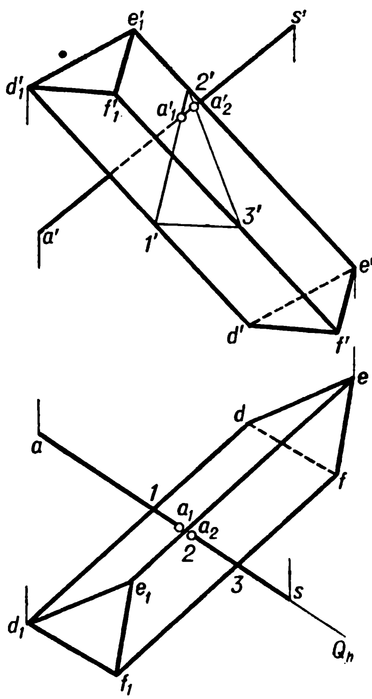

Fig. 281

Figure 280 shows the surface of a triangular prism intersected by a triangular pyramid which is inserted into a hole of the corresponding shape made in the prism. The construction is based on finding the points of intersection of the edges of one polyhedron with the faces of the other. Figure 281 demonstrates the construction of the points A_1 and A_2 at which the edge SA of the pyramid cuts the faces DEE_1D_1 and EFF_1E_1 of the prism. Through the edge SA a horizontal projecting plane Q is passed which intersects (in the horizontal projection) the edges of the prism at points *1*, *2*, *3*. Using these projections, we find the vertical projections $1'$, $2'$, $3'$ of the points of intersection of the plane Q with the edges of the prism. Then we mark points a_1' and a_2' at which $a's'$ intersects the triangle $1'2'3'$. Hence, a_1' and a_2' are the vertical projections of the points at which the edge SA of the pyramid pierces the faces of the prism. The horizontal projections of these points (points a_1 and a_2) are located on the horizontal projection of the edge SA. Proceeding in the same way with the edges SB and SC, we find the points B_1, B_2, C_1, and C_2 (Fig. 280).

Now we find the points of intersection of the edges of the prism with the faces of the pyramid by passing again auxiliary horizontal projecting planes (here, as in the preceding case, we may take advantage of vertical projecting planes). Examining the edge DD_1, we mark the intersection points D_2 and D_3. The edges EE_1 and FF_1 do not pierce the faces of the pyramid.

To avoid mistakes when making numerous auxiliary constructions, it is advisable to tabulate the found intersection points in the following way.

The edge under examination		Faces intersected by the edge	Points of intersection of the edge with the face	No. of place occupied by the point in the general order of joining the points
Pyramid	*SA*	DEE_1D_1	A_1	*1, 6*
		EFF_1E_1	A_2	I
	SB	DEE_1D_1	B_1	*2*
		EFF_1E_1	B_2	II
	SC	DFF_1D_1	C_1	*4*
		EFF_1E_1	C_2	III
Prism	DD_1	*SCB*	D_2	*3*
		SAC	D_3	*5*
	EE_1	None	–	–
	FF_1	None	–	–

In our example two separate polygons are obtained. The above table indicates the order of forming the polygons by Arabic figures *1*, *2*, etc. for one, and Roman figures I, II, etc. for the other. This means that the point a_1' (*1*) should be joined to the point b_1' (*2*), b_1' to d_2' (*3*), d_2' to c_1' (*4*), c_1' to d_3' (*5*), and, finally, d_3' to a_1' (*6*).

In the constructions shown in Fig. 280 and 281 use was made of auxiliary horizontal projecting planes. And though the application of just horizontal- or vertical-projecting planes as auxiliary planes for finding the intersections of a straight line and a plane, or of two planes (and, hence, also in the cases of mutual intersections of polyhedral surfaces) is common practice, since it is a convenient method, still we may come across cases when oblique planes turn out to be preferable as auxiliary planes, since, when used, they lead to a reduced amount of additional constructions. But this should be accompanied by appropriate conditions. An example is given in Fig. 282. Here the bases of both pyramids are contained in a single plane. A straight line is drawn through the vertices of the pyramids and its trace (point M) is found on the plane containing the bases of the pyramids. Any plane drawn through the line ST passes through the vertices of both pyramids and intersects their faces along straight lines (see Fig. 276). The trace of such a plane on the plane containing the bases of the pyramid passes through the point m.

We draw, for instance, a straight line mf and take it for the trace of one of such planes; in Fig. 282 the trace of this plane coincides with the projection mf.

Such a plane cuts the base of the pyramid *ABCS* at points *n* and *r*. Joining these points to the point *s*, we obtain the contour of the section by the taken plane (in which the edge *TF* is contained) and find the projections of the intersection points for the edge *TF*, i.e. points f_1 and f_2. The vertical projections of these points of intersections are determined without difficulties.

Using this method, we examine all the edges of both pyramids to reveal the points required for constructing the line of intersection.

The points of intersection of the sides of the base are determined on the horizontal projection without additional constructions.

The following table contains all necessary data.

The edge under examination	Faces intersected by the edge under examination	Edges intersected by the edge under examination	Points of intersection
TF	*ACS*	–	F_1
	ABS	–	F_2
ET	*CBS*	–	E_1
	ABS	–	E_2
DT	None	–	–
FD	–	*AC*	A_1
	–	*AB*	A_6
DE	–	*BC*	A_4
	–	*AB*	A_5
EF	–	*BC*	A_3
	–	*AC*	A_2
AS	None	–	–
BS	None	–	–
CS	None	–	–

The construction shown in Fig. 282 may be applied even if the base of one pyramid is situated, say, in the *H* plane, and the base of the other is contained in the *V* plane. In this case we have to find the traces of the straight line drawn through the vertices of the pyramids on the horizontal and vertical planes of projection and the horizontal and vertical traces of each auxiliary plane.

The method demonstrated in Fig. 282 for two pyramids may be applied for a mutual intersection of a prism and a pyramid as well. Here a straight line is drawn through the vertex of the pyramid parallel to the edges of the prism. The planes passed through such a line will cut the faces of the prism along straight line parallel to its edges, and the faces of the pyramid along the lines passing through its vertex. In the case of a mutual intersection of two prisms auxiliary secant planes may be taken parallel to the edges of both prisms.

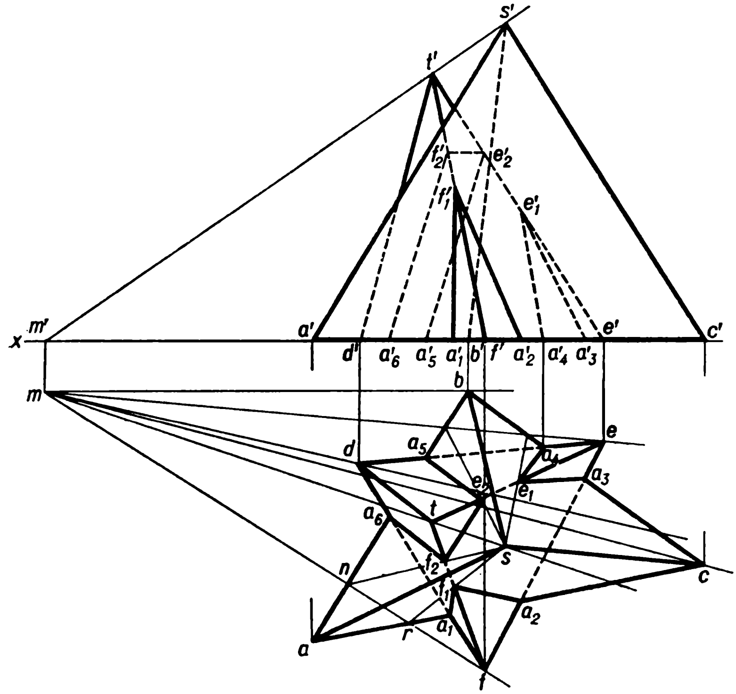

Fig. 282

The method of replacing projection planes may also be practiced if one of the intersecting surfaces is a prism. In this case on obtaining the projections of the polyhedrons on a plane perpendicular to the edges of the prism, we use the faces of prism in this position as secant planes.

Sec. 44. General Methods for Developing Faced Surfaces (Prisms and Pyramids)

A prismatic surface can be developed following one of the two possible schemes.

When using first scheme (Fig. 283) proceed as follows:

(1) intersect the surface by a plane perpendicular to the edges;

(2) determine the length of the segments of the polygonal line obtained as the intersection of the prism by this plane;

(3) develop the polygonal line into a straight line A_0D_0 and on the perpendiculars to the line A_0D_0 drawn through the points A_0, B_0, ... lay off the lengths of the segments of the edges: A_0A, A_0A_1, B_0B, B_0B_1, and so on;

(4) draw the line segments AB, BC, and CD, and also the line segments A_1B_1, B_1C_1, and C_1D_1.

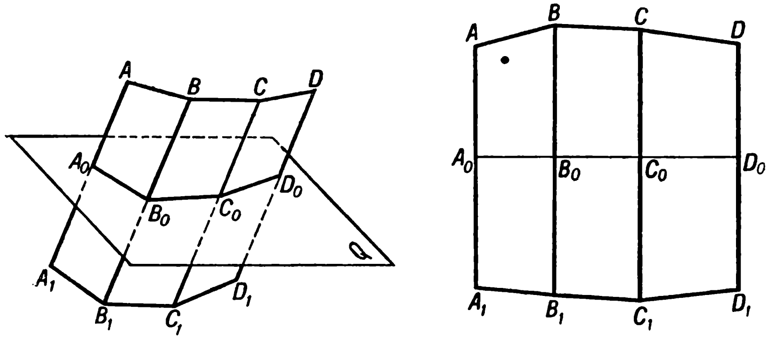

Fig. 283

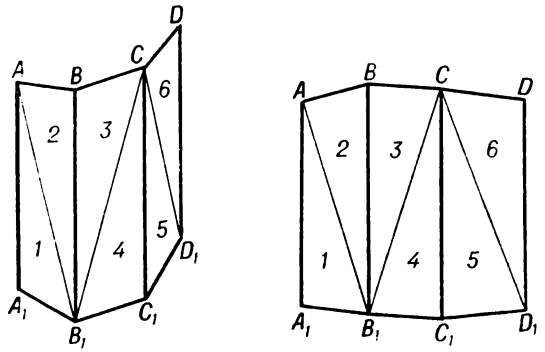

Fig. 284

The second scheme for developing a prismatic surface consists in the following (Fig. 284):

(1) divide the quadrilaterals (the faces) into triangles, using diagonals for this purpose;

(2) determine the lengths of the sides of these triangles;

(3) construct consecutively the triangles *1*, *2*, *3*, etc. in the plane of the drawing.

We can also obtain the required development following the pattern represented in Fig. 287.

Figures 285 and 286 give an example of developing the lateral surface of a prism. The development is constructed here according to the first scheme. All preparatory constructions for the required development are carried out in Fig. 285. First of all, an additional plane of projections is introduced. It is perpendicular to the H plane and parallel to the edges of the prism. To obtain a normal section a plane Q is drawn perpendicular to the edges of the prism. In the system S, H the plane Q is perpendicular to the plane S,

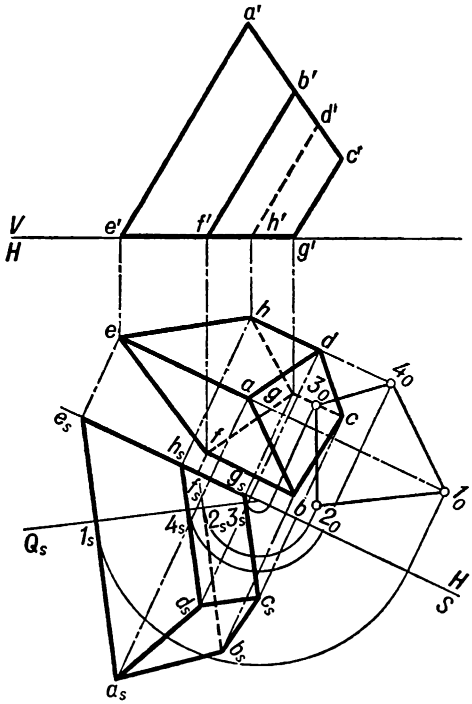

Fig. 285

and therefore the projection of the section figure onto the plane S is found on the trace Q_s. Since the plane Q is perpendicular to the edges of the prism, their projections on the plane S are perpendicular to Q_s, and since the plane S is parallel to the edges, their lengths are equal to the lengths of the line segments a_se_s, b_sf_s, etc. Further, bringing the Q plane into coincidence with the H plane, we determine the true size of the section (the quadrilateral $1_02_03_04_0$).

Figure 286 shows the required development: laid off on a straight line in succesion are line segments $1\text{-}2 = 1_02_0$, $2\text{-}3 = 2_03_0$, etc., and through points 1, 2, etc. perpendiculars are drawn to this line. On them line segments $1A = 1_sa_s$, $1E = 1_se_s$, $2B = 2_sb_s$, etc. are laid off. Finally, polygonal lines $ABCDA$ and $EFGHE$ are constructed.

A different construction is given in Fig. 287. On constructing the projection of the prism on the plane S which is parallel to the edges of the prism, we draw from points e_s, h_s, g_s, f_s, a_s, d_s, c_s, and b_s straight lines perpendicular to e_sa_s. From the point e_s as centre we describe a circular arc of radius equal to eh to intersect the straight line drawn from the point h_s. From the point H_0 thus obtained we strike an arc of radius equal to hg to intersect the line

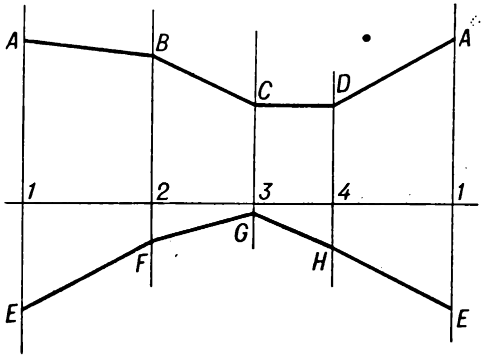

Fig. 286

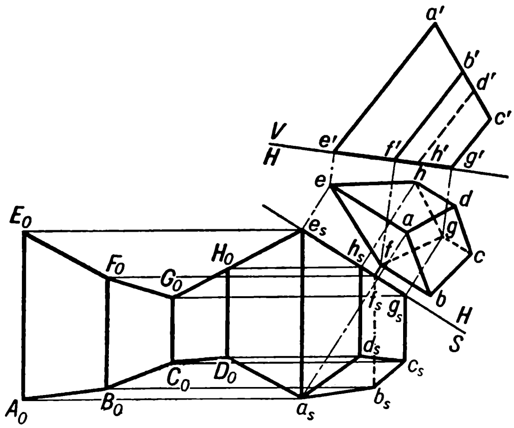

Fig. 287

drawn from the point g_s. As a result, we get a point G_0, and so on ($G_0F_0 = gf$, $F_0E_0 = fe$). From the points H_0, G_0, F_0, E_0 we draw straight lines parallel to e_sa_s to intersect the corresponding lines drawn from the points d_s, c_s, b_s, and a_s.

This alternate version turns out to be expedient, when the lengths of the sides of the base can be taken directly from the drawing.

The lateral surface of a pyramid can be developed using the following scheme:

(1) determine the lengths of the edges and the sides of the base of the pyramid;

(2) construct in the plane of the drawing the triangles representing the faces of the pyramid.

Constructed in Fig. 288 is the development of the lateral surface of a pyramid with the sides of a triangular section drawn on the faces of the

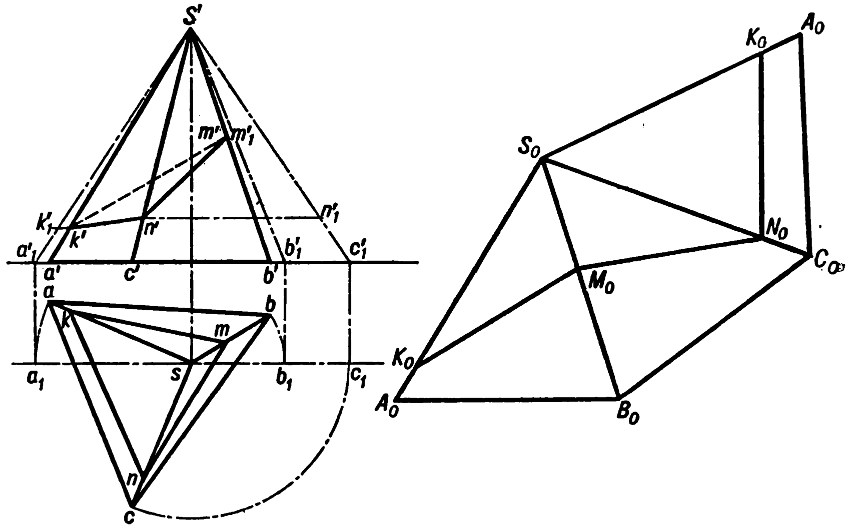

Fig. 288

pyramid. The length of each of the edges is found, and then a triangle $A_0S_0B_0$ is constructed by its sides: the base A_0B_0 is taken equal to the horizontal projection ab, the lateral sides being taken equal to the true lengths of the edges SA and SB (i.e. to the line segments $s'a_1'$ and $s'b_1'$).

Further, on the side S_0B_0 a second triangle is constructed, the other two sides being taken of the following dimensions: the side B_0C_0 is equal to the horizontal projection bc, and the side S_0C_0 to the length of the edge SC (i.e. to the segment $s'c_1'$). A third triangle is constructed in a similar way.

Thus, the development of the lateral surface of the pyramid is obtained. If now on the sides S_0A_0, S_0B_0, and S_0C_0 we lay off line segment S_0K_0, S_0M_0, and S_0N_0 equal to the segments of the respective edges of the pyramid cut by a plane, then we shall get a polygonal line $K_0M_0N_0K_0$ consisting of the sides of the section figure.

QUESTIONS TO SECS. 43-44

1. How do we construct the line of intersection of two faced surfaces?

2. In what case is it expedient to use oblique planes (as auxiliary ones) when determining the intersection of two pyramids, and how are they passed?

3. What schemes do we use for developing surfaces bounding prisms and pyramids?

4. In what case will these developments be complete?

CHAPTER 7

CURVED LINES

Sec. 45. General

*A curved line may be imagined as the path of a moving point in a plane or in space**. Archimedes' spiral and a helix are well known examples of such lines. *A curved line can also be obtained as a result of the mutual intersection of surfaces* (for instance, of two cylindrical surfaces), *or when a surface is cut by a plane* (this is exemplified by an ellipse obtained when the lateral surface of a right circular cylinder is intersected by a plane forming an acute angle with the axis of the cylinder). *In a number of cases a curved line represents a locus of points which satisfy one or more conditions stipulated for this curve* (circles, ellipses, parabolas, and so forth).

A curve is defined by the positions of its points, the latter being defined by their coordinates.

In descriptive geometry curved lines are studied on the basis of their projections. The construction of the projections of lines depends essentially on whether all the points of a given curve lie in one plane or not. If all points of the curve lie in one plane, we have a *plane* curve, examples of which are the circle, ellipse, parabola, hyperbola, Archimedes' spiral, and so on. A curved line that does not lie entirely in one plane is a *space* curve (line of double curvature). These curves are exemplified by helices, lines of intersection of right circular cylinders and cones.

To construct the projections of a curve (in the plane or in space), we have to construct the projections of a number of points belonging to this curve (see Fig. 289). An example of plotting a plane curve point by point was given in Fig. 119 (Sec. 18).

A space curve is projected into a plane curve, and a plane curve is projected in the form of a plane curve or a straight line if the curve is contained in a plane perpendicular to the plane of projection.

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*A curved line must have no rectilinear sections throughout its length.

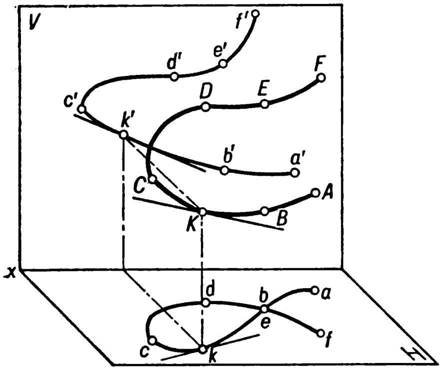

Fig. 289

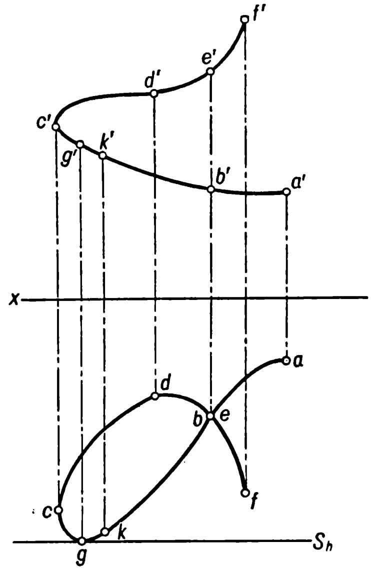

Fig. 290

A line is considered to be *regular* if it is generated in accordance with a geometrical law. If, in addition, a curve is defined in Cartesian coordinates by an algebraic equation, then it is called an *algebraic* curve*. These curves are exemplified by the ellipse whose equation is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. The degree of the equation defines the "order" of a curve: the ellipse is a curve of the second order. The curve representing a projection of a curve of a certain order preserves the same order or turns out to be a curve of a lower order.

A tangent line to a curve is generally projected into a tangent line to the projection of this curve. If, for instance, a tangent line is drawn to a circle contained in a plane forming an acute angle with the plane of projection, then it is projected into a line tangent to the ellipse representing the projection of this circle. Figure 289 demonstrates a space curve, its vertical and horizontal projections, a line tangent to the curve at its point K, and the corresponding projections of this tangent line. The projecting plane passing through the tangent line to a projection of the curve touches the curve in space.

To get a clearer imagination of a curve in space it is advisable, when specifying a plane or a space curve by its projections, to indicate on the projections some points characteristic for the curve itself, or for its location relative to the projection planes. For instance, there may be indicated the points most

*If a curve is defined by a non-algebraic equation, then it belongs to transcendental curves.

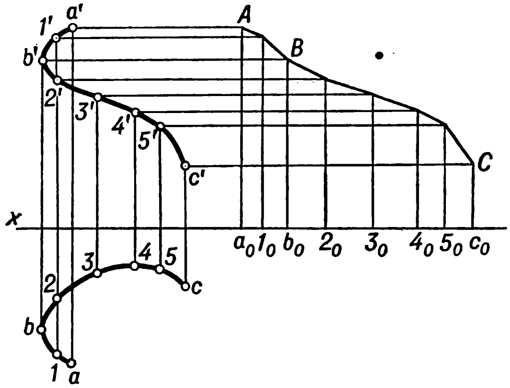

Fig. 291

distant from the projection planes, or nearest to them. This is obtained by passing planes tangent to the curve and parallel to the corresponding planes of projection. Thus, in Fig. 290 the plane *S* which is parallel to the *V* plane makes it possible to determine that the point *G* of the represented space curve is most distant from the *V* plane.

Curvature of a curved line (a plane curve or a space curve) may be constant (along its entire length or over separate segments), or vary at different points of the curve. For instance, curvature of a circle or a helix remains unchanged throughout its length, while curvature of an ellipse, repeated in its quadrants, varies continuously within the limits of one quadrant.

Curvature of a curve is expressed by a number. It characterizes the curve at a given point, or, more precisely, over an infinitely small arc (i.e. within the *neighbourhood* of this point).

The *length* of a certain segment of a plane (or space) curve is determined approximately by replacing a curved line by a polygonal line inscribed in this curve, and measuring the lengths of the segments of this polygonal line (this, of course, does not refer to the curves whose lengths can be determined by rather simple computation*). To reduce the error, the lengths of the segments of the polygonal line should differ but little from the lengths of the arcs of a curve whose chords are these segments. Figure 291 shows how the length of a curve *ABC* is determined: the horizontal projection (curve *abc*) is subdivided into small parts and "developed" into a straight line on the x-axis so that the segments $a_0 1_0$, $1_0 b_0$, etc. are respectively equal to the chords $a1$, $1b$, etc. At points $a_0 1$, $_0$, etc. perpendiculars are drawn to the x-axis, and the z-coordinates of the points on the curve are laid off on them. As a result, we get a polygonal line whose length may be approximately taken for the length of the curve *ABC*.

*For instance, the circle, a turn of the helix (see Sec. 48).

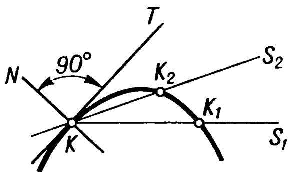

Fig. 292

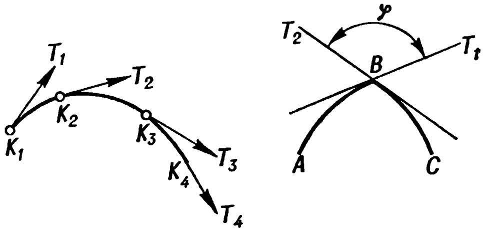

Fig. 293

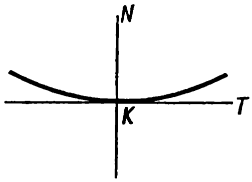

Fig. 294

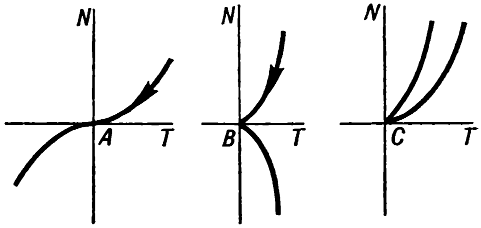

Fig. 295

Sec. 46. Plane Curves

Revolving a secant line KS_1 (Fig. 292) about the axis K so that the point K_1 tends to the point K, we obtain a limiting position KT, i.e. the position of the tangent line to the curve at its point K. The tangent line indicates the direction of motion of the point generating the curve. The direction of the tangent line at a certain point of a curve is called the *direction of the curve at this point.*

Drawing at the point K a straight line KN perpendicular to KT, we obtain a normal* to the curve at its point K. The normal to a circle coincides with the direction of its radius. The construction of a normal to an ellipse is shown in Sec. 21.

The curve at point K (Fig. 292) is *smooth*: it has one tangent at this point. If a curve consists only of such points, then it is a *smooth curve* (Fig. 293, left). But a curve may have points (see Fig. 293, right) at which there are two tangent lines intersecting at an angle not equal to 180°. Such points are called *salient points* of a curve, and the latter is not smooth at such a point, as if two curves (AB and BC) intersect at such a point at a certain angle. If this angle (φ) turns out to be equal to 180°, then the curves AB and BC will get in touch with each other, and either of them will appear smooth at point B. Touching curves have one and the same tangent line at their common point, and the normals to the curves at this point are located on a single perpendicular.

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*From the Latin 'normalis' meaning 'rectilinear', 'straight'.

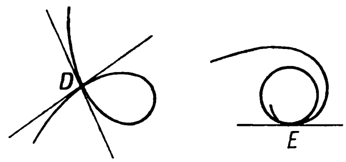

Fig. 296

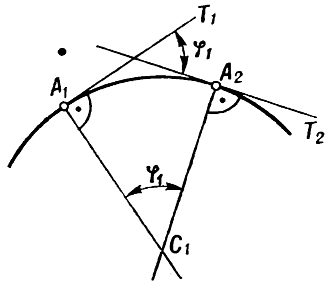

Fig. 297

In Figure 294 a tangent KT and a normal KN are drawn at point K on a curve. If at all points of the curve the same arrangement is repeated (with respect to the tangent and the normal within the neighbourhood under consideration), then the curve is *convex* and its points are *regular* (for instance, the ellipse).

Figure 295 illustrates a number of irregular, or so-called singular points: a *point of inflection* (A) at which the curve intersects the tangent line, and two *cusps* (B and C) at which the curve has a "beak" and a tangent common for both branches (point B is called a *cusp of the first kind*, and C a *cusp of the second kind*). Hence, a cusp (or spinode) is a double point at which the two tangents to the curve are coincident and the direction of motion of the point describing the curve is reversed. A cusp of the first kind (or simple cusp) is a cusp in which there is a branch of the curve on each side of the double tangent in the neighbourhood of the point of tangency. A cusp of the second kind is a cusp for which the two branches of the curve lie on the same side of the tangent in the neighbourhood of the point of tangency. A double cusp is the same as a point of osculation (see below).

Figure 296 represents two more singular points: a *node* (D), and a *point of osculation* (E). A node is a point at which two parts of a curve cross and have different tangents (point D on the left-hand curve). A point of osculation is a point on a curve at which two branches have a common tangent and each branch extends in both directions of the tangent. It is also called a *tacnode* and a *double cusp* (point E on the right-hand curve).

All mentioned singular points may occur on the projections of plane curves. To judge the character of its points, it is sufficient for a plane curve to have only one projection (provided this projection is not a straight line), since any singularity of this projection represents the same singularity of the plane curve itself.

The curvature of a plane curve at some of its points A_1 (Fig. 297) *is regarded as the limit to which tends the ratio of the angle between the tangents drawn at two neighbouring points* A_1 *and* A_2 *of the curve to the arc* A_1A_2 *as the point* A_2 *tends to* A_1:

$$\lim \frac{\varphi_1}{\overset{\frown}{A_1A_2}} = k.$$

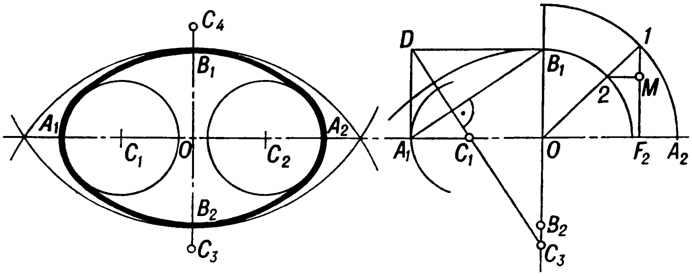

Fig. 298

Thus, *the curvature of a curve at its point A is defined as the limiting value of the angle φ_1 to the arc A_1A_2*. Or in other words, the absolute value of the ratio of the change in the angle of inclination of the tangent line along a given arc to the length of the arc is the average curvature along the arc. The limit of the average curvature as the length of the arc approaches zero is the curvature. We shall denote curvature by the lower-case letter k.

Obviously, the angle φ may also be represented as the angle between the normals to the curve at the points A_1 and A_2.

If we imagine a circle passing through the point A_1 (Fig. 297) and two adjacent to it points on the curve tending to the point A_1, then the circle will come to its limiting position when the point of intersection of the normals (C_1) occupies its limiting position thus determining a certain radius C_1A_1. Moreover, the circle will get in touch with the curve at point A_1. In this position the circle and the curve will have a common tangent and a common normal on which the centre of the touching circle is located.

The following terms are associated with the notion of curvature: the *circle of curvature of a curve at a given point*; the *centre of curvature* (or the centre of the circle of curvature); the *radius of curvature* (or the radius of the circle of curvature). *The curvature of a curve at any of its points is the reciprocal of the radius of curvature*: $k = \frac{1}{r}$. Obviously, for a circle, the osculating circle at any point of contact has a radius equal to the radius of the given circle. Hence, the *curvature of a circle at any of its points is the reciprocal of the radius of this circle*: $k_{\text{circle}} = \frac{1}{R}$. The greater R, the less is k.

Let us define the above terms: the circle tangent to a curve on the concave side and having the same curvature at the point of tangency is called the circle of curvature of the curve (at that point). Its radius is the numerical value of the radius of curvature, and its centre is the centre of curvature.

Figure 298 (left) shows an ellipse with the centres of curvature at vertices A_1 and A_2 located on its major axis, and at vertices B_1 and B_2 on its minor axis. To determine the positions of the centres of curvature, we take advantage of the known formulas for the radii of curvature at the vertices of the ellipse: $r_1 = \frac{b^2}{a}$ (for the vertices A_1 and A_2), and $r_2 = \frac{a^2}{b}$ (for the

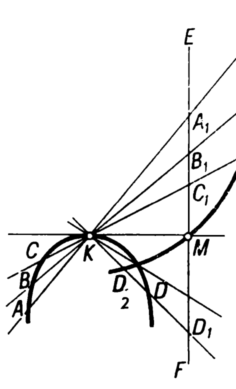

Fig. 299

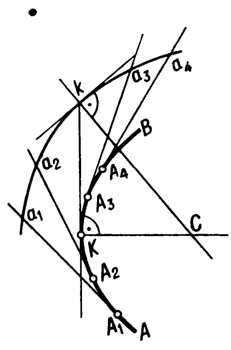

Fig. 300

vertices B_1 and B_2), where a is the semimajor axis and b is the semiminor axis of the ellipse.

The right-hand drawing (Fig. 298) illustrates the construction of the centres of curvature C_1 and C_3, and determination of the radii of curvature at the vertices A_1 and B_1. Using the given semi-axes OA_1 and OB_1, we construct a rectangle OB_1DA_1 and its diagonal A_1B_1. Then from point D a perpendicular is dropped on this diagonal to intersect the major axis at point C_1 and the extension of the minor axis at point C_3. If circular arcs are drawn—one from the point C_1 of radius C_1A_2, and the other from the point C_3 of radius C_3B_1, then a gap is obtained between these arcs, where, with the aid of a French curve, an arc is drawn touching the two circular arcs. To draw this arc more accurately, it is advisable to find a point belonging to an ellipse as it is shown on the same drawing for the point M on the straight line drawn through the focus F_2 perpendicular to the axis A_1A_2 of the ellipse. The construction is accomplished in the following succession: the focus F_2 (see Sec. 21), arcs of radii OA_1 and OB_1, perpendicular to A_1A_2 erected at the point F_2 to intersect the arc at point *1*, radius *0-1*, and through point *2* a straight line parallel to OA_2. Using the found points C_1 and C_3, we can find two more centres, and with the aid of the point M—three more points needed for constructing the remaining part of the curve. This combined line approximates the ellipse rather closely.

What plane curve has a constant curvature? The answer is: the circle (see above: $k_{\text{circle}} = \frac{1}{R}$, where R is the radius of the circle). If the straight line is taken for the circle at $R = \infty$, then here the curvature is also constant, i.e. $k = 0$.

An approximate construction of a tangent and a normal to a smooth curve at its point K is shown in Fig. 299.

We begin the construction with drawing an auxiliary straight line EF roughly perpendicular to the supposed direction of the tangent to the curve

ABCD. Then, through the point K we draw several straight lines intersecting the curve *ABCD* and the straight line *EF*. If now we lay off $A_1A_2 = AK$, $B_1B_2 = BK$, $C_1C_2 = CK$, $D_1D_2 = KD$, etc. and through the points A_2, B_2, C_2, D_2, ... draw a smooth curve, then its intersection with the line *EF* will yield a point M which is a second point defining the tangent to the curve *ABCD* at the point K*.

Figure 300 illustrates an approximate construction of the centre of curvature at some point K of a curved line.

We take on the curve several points A_1, A_2, ... close to the point K and draw tangents to the curve at these points (including the point K). Then we lay off arbitrary, but equal to one another, line segments A_1a_1, A_2a_2, Kk, ..., and draw a smooth curve through the points a_1, a_2, k, ... thus obtained. The intersection of the normals at the points K and k yields a point C which is the required centre of curvature, the radius of curvature r being equal to CK. Whence the curvature at point K is determined, it is equal to $\frac{1}{r}$.

If the centres of curvature are constructed at a number of points of a given curve, then through these centres we may pass a curve called the *evolute* which is the *locus of the centres of curvature of the given curve.* The given curve is then called the involute with respect to its evolute*. For instance, in the curve called the involute of a circle, the centres of curvature at different points of this curve are located on a circle which is just the evolute with respect to the given involute.

Sec. 47. Space Curves

Many things considered with respect to plane curves are applicable to space curves. For instance, the tangent to a space curve is also obtained from the secant KS_1 (see Fig. 292) when the point K_1 merges with K. A space curve may also have various points: regular points, points of inflection, cusps, etc. But while for a plane curve we could draw through the point K (Fig. 292) only one perpendicular KN (a normal) to the tangent KT, in the case of a space curve we have an infinite number of such perpendiculars at the point of tangency—the fact which leads to the notion of a *normal plane.* Furthermore, for a plane curve one projection suffices to judge on the character of its points, for a space curve we need two projections for the same purpose. For instance, examining the horizontal and vertical projections represented in Figs. 289 and 290, we come to a conclusion that there is no double point on the curve in space, despite the fact that the horizontal projection of the curve has such a point. The *same as for a plane curve, the tangent to a curve in space is projected into a tangent to the projection of*

*The curve $A_2B_2C_2D_2$ is an example of the so-called error curve.

*From the Latin 'evoluta' meaning 'developed'.

this curve (Fig. 289). A projecting plane passed through a tangent to a projection of the curve touches the curve in space.

A plane curve lies entirely in a plane. As far as a space curve is concerned, we may speak only of a plane approaching the curve most closely at the point under consideration. Such plane is said to be an *osculating plane.* Suppose Fig. 292 represents a segment of a space curve (but not a plane curve). Then three points (K, K_1, and K_2) of this curve define a plane. The limiting position of this plane, when the secant KS_2 becomes a tangent at point K and the third point approaches the point of tangency most closely, defines the osculating plane of the space curve at the point K. Near this point we may consider the curve as lying in the osculating plane.

The osculating and the normal planes are mutually perpendicular. This follows from the fact that the osculating plane contains the tangent to the curve.

The intersection of a normal and an osculating planes yields one of the normals called the *principal normal.* A normal perpendicular to the osculating plane is called the *binormal.*

Added to the osculating and normal planes is a third plane which is perpendicular to them. It passes through the tangent and binormal and is termed the *rectifying plane.*

These three planes forming a trihedron, are used as the coordinate planes when considering a curve at a given point. The position of the trihedron depends on the position of the point on the curve.

Analogously to the centre of curvature for a plane curve as a limiting position of the point of intersection of two normals (Fig. 297), we obtain the *axis of curvature of a space curve* as the limiting position of the line of intersection of neighbouring normal planes. In this limiting position the *axis of curvature is parallel to the binormal. Intersecting the principal normal, the axis of curvature yields the centre of curvature,* whence we get the radius of curvature as the distance from this centre to the point of the curve under consideration. The same as for a plane curve, the curvature of a space curve is the reciprocal of the radius of curvature. If we imagine the limiting approach of three adjacent points on a space curve and the limiting position of a circle drawn through them, then we obtain the circle of curvature in the osculating plane, its centre being the centre of curvature and its radius the radius of curvature. This is the first curvature of a space curve.

If instead of the angle between the tangents, as it happened in the case of plane curves, and the ratio of this angle and the length of the arc between the points of tangency we take the angle between the osculating planes (it is equal to the angle between the binormals) and divide this angle by the length of the arc between the points of the space curve under consideration, then the limiting value of this ratio represents the so-called curvature of torsion or the second curvature of a space curve. Let us remember that space curves are also called curves of double curvature.

If the tangents to a space curve at all its points are equally inclined to some plane, then such lines are termed the lines of equal inclination (or slope).

QUESTIONS TO SECS. 45-47

1. What is the difference between a plane curve and a space curve?
2. Into what is a space curve projected?
3. Into what is a plane curve projected?
4. Into what is a tangent to a curve projected?
5. How do we determine the length of some portion of a curve?
6. What is the tangent to a curved line?
7. What is the normal at some point of a plane curve?
8. What determines the smoothness of a plane curve?
9. What plane curves are called osculating curves?
10. What is a convex plane curve?
11. How many projections are sufficient to judge on the character of points of a plane curve?
12. What is the curvature of a plane curve at some of its points?
13. What is the curvature of the circle equal to?
14. How do we construct a combined curve resembling the ellipse if the axes of the ellipse are given?
15. How do we construct the tangent and the normal to a smooth curve at one of its points?
16. How many projections are sufficient to judge on the character of points of a space curve?
17. What planes are called normal, osculating, and rectifying at some point of a space curve?
18. What are the principal normal and binormal at some point of a space curve?
19. What are the first and the second curvatures of a space curve?
20. What is understood under "a curve of double curvature"?
21. In what case is a space curve called the curved line of equal slope?

Sec. 48. Cylindrical and Conical Helices

The *cylindrical helix* represents a space curve of equal slope. A helix can be obtained on a cylinder in the following way: clamp a cylindrical rod in the chuck of a lathe and set it into uniform rotary motion; bring a thread tool into contact with the cylinder and set it into uniform forward motion along the cylinder axis; the point of the tool will draw a helix on the surface of the cylinder.

Figure 301 illustrates how a cylindrical helix is generated by a point A in motion along the generating line (generatrix) EC of a right circular cylinder rotating about its axis. The path traversed by the point along the generatrix is all the time proportional to the angle of turn of the cylinder. Shown in

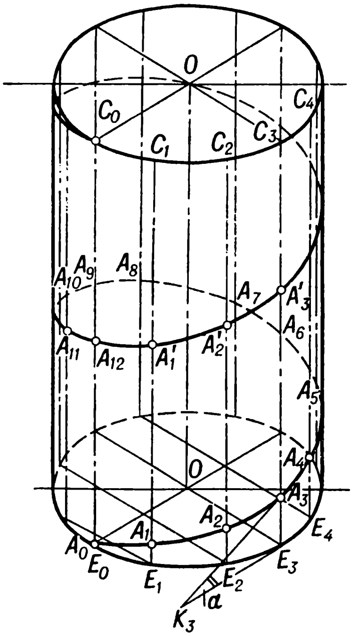

Fig. 301

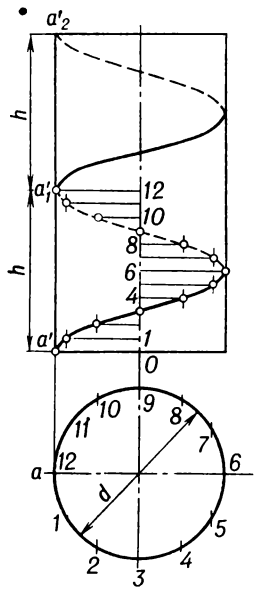

Fig. 302

the figure are several positions of this generatix: E_0C_0, E_1C_1, . . . ; here the arcs E_0E_1, E_1E_2, . . . are of equal length, each of them being equal to $\pi d/n$, where d is the diameter of the cylinder, and n is the number of divisions (in Fig. 301 $n = 12$). The initial position of the point is denoted by A_0, the subsequent positions by A_1, A_2, etc.

If during the displacement of the generatrix from the position E_0C_0 to the position E_1C_1 the point occupies the position A_1, then the line segment E_1A_1 will determine the distance covered by the point along the generatrix. In the subsequent position of the generatrix (E_2C_2) the point will be found at a height $E_2A_2 = 2E_1A_1$, and so on. When the generatrix accomplishes a complete revolution, the point will cover the distance $E_0A_{12} = 12E_1A_1$.

As the generatrix continues its rotation, the point A begins forming a *second turn* (or *revolution*) *of the helix*, occupying the consecutive positions A'_1, A'_2, etc.

The distance between the points A_0 and A_{12} is called the *pitch* (or lead) *of the cylindrical helix*. The pitch may be chosen depending on specific conditions.

The distance between the point A and the axis OO is called the *radius of the helix*, and the axis OO the *axis of the helix*. The radius of a helix is equal to half the diameter of a right circular cylinder on whose lateral surface the helix is located. Two quantities—the diameter of the cylinder and the pitch

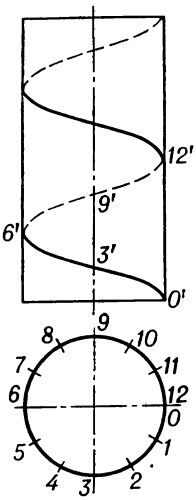

Fig. 303

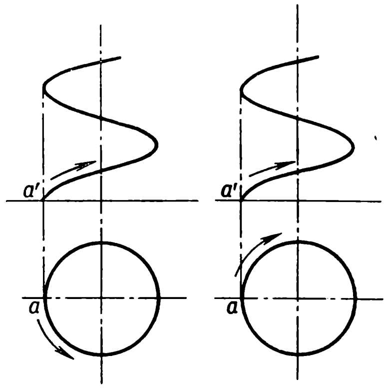

Fig. 304

of the helix—are the *parameters* which define a cylindrical helix on the lateral surface of a right circular cylinder.

Figure 302 gives an example of constructing the projections of a cylindrical helix. First the projections of a right circular cylinder are constructed. Then both the circle (which is the horizontal projection of the cylinder) and the lead (the line segment h laid off along the axis of the cylinder on the vertical projection) are divided into equal number (n) of parts (in our case $n = 12$). The initial position of the point A is indicated by the projections a' and a.

Since the axis of the cylinder is perpendicular to the H plane, the horizontal projection of the helix coincides with the circle representing the horizontal projection of the cylinder surface. The construction of the vertical projection of the helix is obvious from Fig. 302, bearing in mind that a point generating this line performs two motions: a uniform motion along a straight line and, at the same time, a uniform rotary motion about the axis parallel to this line.

The projection on a plane parallel to the cylinder axis, in our case the vertical projection of a cylindrical helix, is similar to a *sinusoid.*

We see on the vertical projection of the helix (Fig. 302) a *rise from left to right*, or a *slope to the left* on the front (visible) side of the cylinder. If the axis of the cylinder is arranged horizontally, then the helix rises leftwards and slopes rightwards. This is a *right-hand helix.* A *left-hand helix* is shown in Fig. 303. Here the line rises from right to left on the front (visible) side of the cylinder on the vertical projection of the helix, the slope being directed rightwards. If the axis of the cylinder is brought to a horizontal position, then the helix rises rightwards and slopes leftwards.

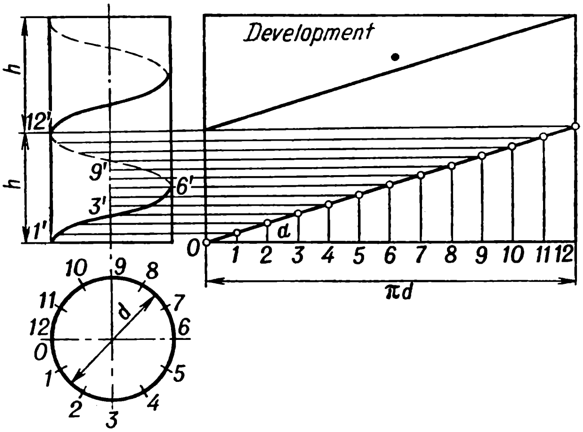

Fig. 305

If a helix is represented without the cylinder and without the projections of the points of division, then its direction is indicated by an inscription or an arrow as in Fig. 304 in which the left-hand drawing represents a right-hand helix, and the right-hand drawing a left-hand helix*.

The development of a complete turn of a cylindrical helix is given in Fig. 305. Upon development of the cylindrical surface on a plane, each turn of the helix develops into a straight line. This follows from the way a cylindrical helix is generated: since the circumference of the cylinder base has been divided into a given number of equal parts and the lead of the helix has also been divided into the same number of equal parts, the development of the helix within the limits of its lead (i.e. its complete turn) may be considered as the locus of points for each of which the ordinate is directly proportional to the abscissa, that is $y = kx$, which is the equation of a straight line.

The tangents to the helix coincide on the development with the straight line into which a turn of the helix is developed.

Figure 305 represents two complete turns (two leads) of a cylindrical helix developed into two parallel lines at an angle α to the straight line representing the developed circumference of the base of the cylinder. The slope of a cylindrical helix is expressed by the formula

$$\tan \alpha = \frac{h}{\pi d},$$

where h is the lead of the helix, and d is the diameter of the cylinder; α is called the helix angle.

*A cylindrical helix is well illustrated by a coiled cylindrical spring, a thread on bolts, screws, pins, and a cylindrical worm.

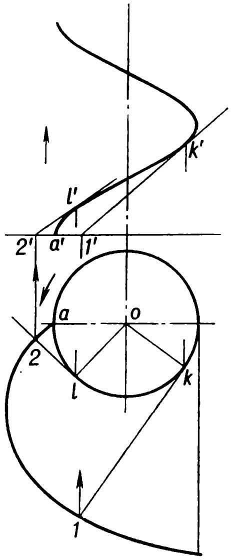

Fig. 306

The length of one turn of a cylindrical helix is

$$L = \sqrt{h^2 + (\pi d)^2}.$$

For one and the same d the magnitude of the angle α depends only on the lead of the helix. To obtain a small angle of slope, a small lead should be taken, and vice versa. If the lead remains unchanged for cylinders of different diameters, then the helix angle is inversely proportional to the diameter of the cylinder.

We may construct the model of a cylindrical helix in the following way: take a rectangle with a diagonal drawn on it and roll it up into a right circular cylinder. In this case the diagonal of the rectangle will form one complete turn of the helix. Obviously, the helix is the shortest distance between two points on the surface of a circular cylinder, i.e. the *geodetic line of this surface.*

Indeed, we can draw a great number of lines between two points on the surface of such a cylinder, one of these lines representing the shortest distance between the points. Upon development of the surface, such a line develops into a straight line. This property is inherent in geodetic lines (geodesics).

Let us consider the following property of a cylindrical helix. Suppose (Fig. 301) that to the helix, at any of its points, say A_3, a tangent is drawn intersecting the H plane at point K_3.

The angle between the helix and any element of the cylinder is expressed by the angle between this element and the tangent to the helix drawn at a point common for the helix and the element. The development shown in Fig. 305 indicates that between the given helix and the element of the cylinder a constant angle is obtained, i.e. *all the tangents to the helix are equally inclined to the elements of the cylinder and intersect the H plane at one and the same angle α.* The same angle was obtained between the developed helix and circumference of the cylinder base.

When developing the lateral surface of a cylinder with a helix on it, the element $A_0A_3E_3$ (Fig. 301) takes the form of a right-angled triangle $K_3A_3E_3$ in which K_3A_3 is the tangent to the helix at point A_3, and K_3E_3 is the projection of the tangent on the plane containing the cylinder base, i.e. the tangent to the circumference of its base. Hence it follows that the point K_3 belongs to the involute of the circle, since the tangents to all points of a cylindrical helix have traces on the plane of the cylinder base which form the involute of the circle of the base of this cylinder.

Let us take advantage of this property for constructing a tangent to a cylindrical helix at any of its points. To the helix represented in Fig.306 the tangent is constructed at point K. First of all the horizontal projection of

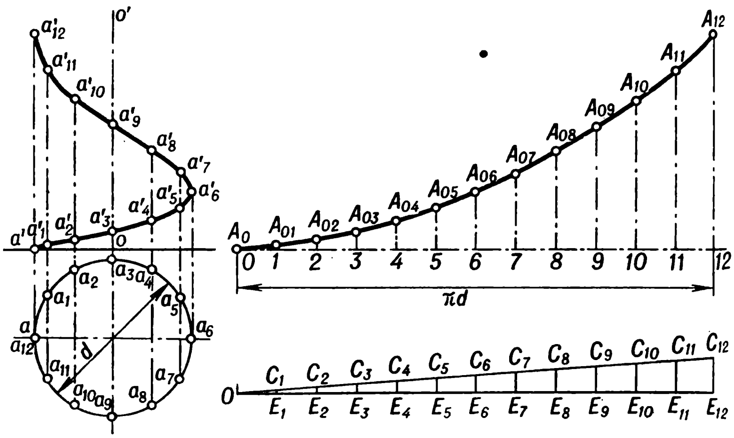

Fig. 307

the tangent (line segment *kl*) is drawn perpendicular to *ok*. From point *l* on the involute the projection *l'* is found, and then the vertical projection of the tangent (the line *l'k'*) may be drawn. The construction is repeated for the point *L*.

We may construct on the cylindrical surface a curved line generated in the same way as the helix with the only difference consisting in that the point performs not uniform but *variable* motion along the generatrix obeying some law. Such curves are sometimes called *helices with a variable lead.*

The construction is given in Fig. 307 for a point in uniformly accelerated motion along the generatrix. The corresponding displacements of the point are given for each of the twelve positions of the generatrix shown in the drawing. For instance, during displacement from the eighth to the ninth position the point covers a distance equal to the segment C_9E_9.

The development of the constructed curve is also given in the same figure. Obviously, the helix angle is variable this time.

If a point uniformly displaces along the generatrix of a right circular cone, and the generatrix rotates at the same time about the axis of the cone with an angular speed, then the path traversed by the point is a *conical helix**. Its projections are given in Fig. 308. The displacements of the point are proportional to the angular displacements of this generatrix. Indicated in the drawing are twelve positions of the generatrix with the corresponding positions of the point on them. The distance between the corresponding

*A conical helix is well illustrated, for instance, by a coiled conical spring or a conical thread.

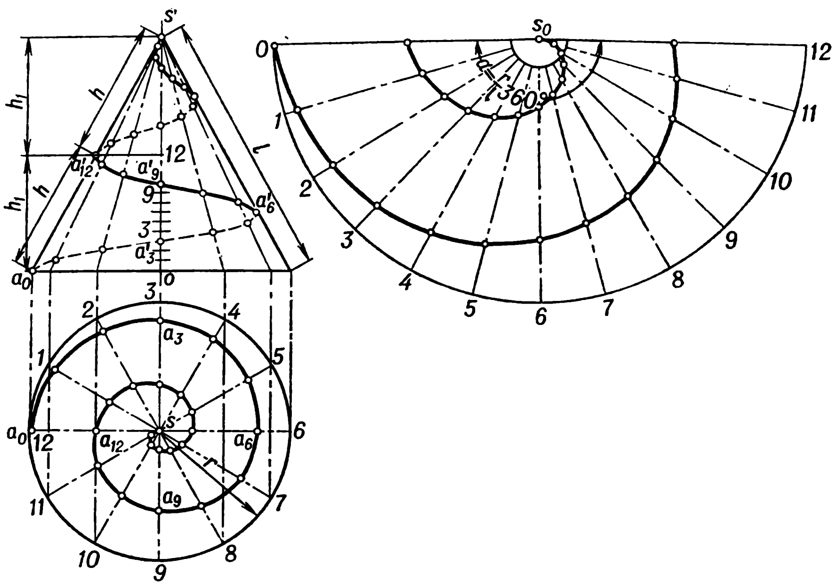

Fig. 308

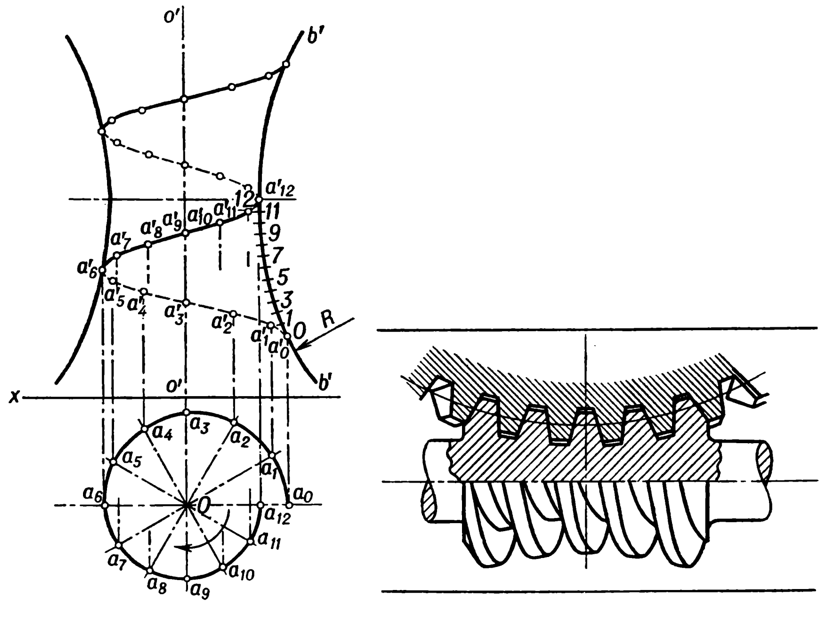

Fig. 309

points of adjacent turns ($A_0A_{12} = h$) measured along the generatrix is called the *lead (pitch)* of the conical helix*.

The projection of a conical helix on a plane parallel to the axis of the cone (in our case the vertical projection) represents a *sinusoid* with a decreasing length of wave, the projection on a plane perpendicular to the axis of the cone (in our case the horizontal projection) being the *spiral of Archimedes.*

Upon development of the lateral surface of the cone (the right-hand drawing of Fig. 308), the helix is developed also into a spiral of Archimedes, since to a uniform angular displacement of the radius on the developed surface of the cone there corresponds a uniform displacement of the point along this radius. The mentioned figure represents the development of two complete turns of the conical helix.

A helix may be constructed not only on a cylindrical or a conical surfaces. This may be exemplified by a helix on the surface generated by an arc *BB* about the axis *OO*, i.e. on the surface of a torus (see Fig. 309, left). A similar helix is sometimes used in cone worms (Fig. 308, right).

QUESTIONS TO SEC. 48

1. How are a cylindrical and a conical helices generated?
2. What is the lead (pitch) of the helix (cylindrical and conical)?
3. What form have the projections of a cylindrical and a conical helices on the planes parallel to the helix axis and perpendicular to this axis?
4. How do we recognize what helix (a right-hand or a left-hand) is marked on the surface of a cylindrical and a conical rods? How is the direction of run indicated if only a helix is represented?
5. Into what line is each turn of a cylindrical helix developed? The same question about a conical helix.
6. How is the slope of a cylindrical helix expressed?
7. What line is formed on a plane perpendicular to the axis of a cylindrical helix by constructing the traces of the tangents to this helix?

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*The lead of a conical helix is sometimes considered as measured along its axis. The line segment h_1 (Fig. 308) is the projection of the lead h measured along the generatrix on the axis of the helix. To the division of h into n equal parts there corresponds the division of h_1 into the same number of equal parts, and vice versa.

CHAPTER 8

CURVED SURFACES

Sec. 49. General

1. A surface may be thought of as a common part of two adjacent domains of space. In descriptive geometry the surface is defined as the trace of a moving line or another surface. The idea of the surface as the totality of all consecutive positions of a line moving in space is convenient for graphical representations. Of course, when representing a surface, we usually confine ourselves to showing this line only in some of its positions.

The idea of generating a surface by continuous motion allows us to term such surfaces *kinematic.*

When depicting a surface, it is more convenient to regard it as a locus not of points but of certain lines that satisfy definite conditions. These lines, which may be straight or curved, are called *generating lines (generatrices).* To construct the projections of a surface one must know the type of generatrix and also its mode of motion.

The same surface may be generated in a number of ways. For instance, the surface of a right circular cone may be generated by the revolution of a straight-line generatrix about the axis intersecting it or by the translation motion of a constantly deforming circle whose centre moves along the axis of the cone, while the plane of the circle is perpendicular to the axis. Of the great diversity of methods of generating surfaces, we must select those that combine a simple shaped generating line with uncomplicated kinematics.

Thus, a kinematic surface represents a locus of lines moving in space according to a certain law.

A surface generated according to such a law is called *regular*, as distinguished from *irregular* surfaces.

2. A surface which can be generated by a straight line is called the *ruled surface.* Hence, the ruled surface represents a locus of straight lines. A surface which may be generated only by a curved line will be called the *double-curved surface.*

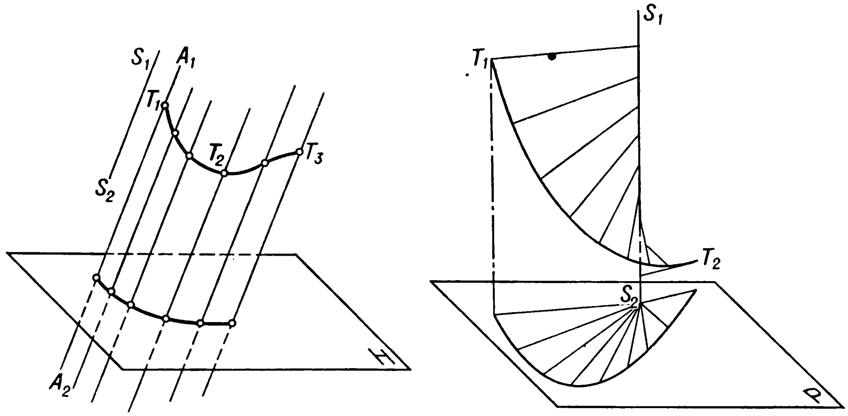

Fig. 310

Examples of ruled surfaces are given in Fig. 310. The surface depicted at the left is generated by a straight line A_1A_2 which, remaining all the time parallel to S_1S_2, slides along a fixed curved line $T_1T_2T_3$ called the *directrix.*

Obviously, the same surface is generated if the curve $T_1T_2T_3$ is regarded as the generatrix whose points are moving along straight lines parallel to the directrix S_1S_2. Of course, in all its positions the curve must obey the conditions of equality and parallelism of curves, and of mutual parallelism of the tangents drawn to the curve at one and the same point in the consecutive positions.

The surface represented to the right is generated by a straight line which, remaining all the time parallel to the plane P, slides along two fixed *directrices:* a straight line S_1S_2 and a curved line T_1T_2.

A double-curved surface may be exemplified by a sphere (or rather a spherical surface).

3. As it was mentioned in Item 1, one and the same surface may be generated by different lines and according to different conditions to be obeyed by the generatrix during its displacement. For instance, the lateral surface of a right circular cylinder (Fig. 311) may be considered as a result of a definite displacement of the generatrix (a straight line A_1A_2), or as a result of the displacement of a circle whose centre moves along the straight line O_1O_2, and the plane defined by this circle is perpendicular to O_1O_2. Let us now consider the curve $T_1T_2T_3$ located on the same cylinder. All of its points are equidistant from the cylinder axis O_1O_2. We may regard the lateral surface of this cylinder as generated by the curved line $T_1T_2T_3$ revolving about the axis O_1O_2.

In general, there is a variety of laws for generating a certain surface. It is desirable to select those laws and shapes of the generating lines which are most simple and convenient both for representing the required surface and solving the problems associated with it. If we imagine the totality of

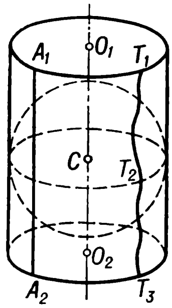

Fig. 311

rectilinear generatrices and the totality of generating circles (Fig. 311), then each line of one totality (of one "family" of lines) will intersect all lines of the other totality (of the other "family" of lines). As a result, the *network* of the given surface is obtained. Other surfaces may be thought of in the same way.

4. The lateral surface of the cylinder shown in Fig. 311 can be generated by displacing a sphere whose centre C moves along the axis O_1O_2. Here we have not a generating line but a generating surface (a sphere). The surface thus obtained (the lateral surface of the cylinder) envelops the generating surface (the sphere) at all its positions, both surfaces contacting each other along a circle in each position of the sphere.

If the centre of the sphere moved along a curve, then, of course, we would obtain another enveloping surface (see Fig. 349).

Hence, we may regard the generation of a surface as a result of displacement of a generating surface which remains unchanged or continuously varies according to a certain law during its motion.

5. Some curved surfaces can be developed so that they coincide completely (with all their points) with a plane without stretching or shrinking. In this case each point on the development corresponds to a single point of a surface; straight lines belonging to a surface remain straight; line segments preserve their lengths; an angle formed by lines on a surface remains equal to an angle between the corresponding lines on the development; the area of a closed domain on a surface retains its magnitude within the corresponding closed domain on the development*.

Such surfaces will be called *developable. They comprise only ruled surfaces in which adjacent rectilinear generatrices are parallel or intersect, or are tangent to sphere curve.*

All double-curved surfaces and the ruled surfaces which cannot be developed into a plane are called *nondevelopable* (or *warped*) surfaces.

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*The angle between two intersecting curved lines is defined as the angle between the tangents to these lines at the point of their intersection.

Sec. 50. A Review of Some Curved Surfaces, Their Specification and Representation in Drawings

To specify a surface in a drawing means to indicate the conditions enabling us to construct each point of this surface. For a surface to be specified, it is sufficient to have the projections of its directrix and adequate information on the method for constructing the generatrix passing through any point of the directrix**. But if it is desirable to make the representation more obvious and expressive, then it is advisable to draw also the outline of the surface, several positions of the generatrix, most important lines and points on the surface, etc.

A. Developable Ruled Surfaces

1. Cylindrical and Conical Surfaces. *A cylindrical surface* is generated by a straight line which is parallel in all its positions to a given straight line and passes in succession through all points of a curved *directing line* (see Fig. 310, left).

A *conical surface* is generated by a straight line passing through a fixed point and through all the points (in succession) of a curved *directing* line (Fig. 312). The fixed point S is called the *vertex* of a conical surface. If this point is removed to infinity, then a conical surface turns into cylindrical.

Cylindrical and conical surfaces may intersect a plane of projection. The line thus obtained is called the *trace of a surface* on a given plane of projection.

Figure 313 represents a cylindrical surface (left) specified by a curved directrix $A_1B_1C_1$ and the direction ST for the generatrix, and a conical surface (right) specified by a curved directrix $K_1M_1N_1$ and the vertex S. In both cases constructed on the H plane are the traces of the surfaces, i.e. the lines passing through the horizontal traces of the generating elements of the given surface (the curves $a'b'c'$, abc and $k'm'n'$, kmn).

A cylindrical surface may be specified by its trace on the H plane and the direction of its generatrix, a conical surface by its trace on the H plane and its vertex. Taking a point on the trace, we can construct the corresponding element of the surface.

To construct the contour of a cylindrical or a conical surface, "boundary elements" should be marked on either plane of projection which contain the domain enclosing the projection of the surface. For instance, marked in the left-hand drawing of Fig. 314 on the trace of a cylindrical surface are the points through which the projections of the boundary elements pass: a', a and b', b for the vertical projection; c', c and d', d for the horizontal projection. These boundaries, together with the break lines, determine the contours of the projections and separate the surface into visible and in-

*The directrix is often specified by the line of intersection of a given surface with the H plane.

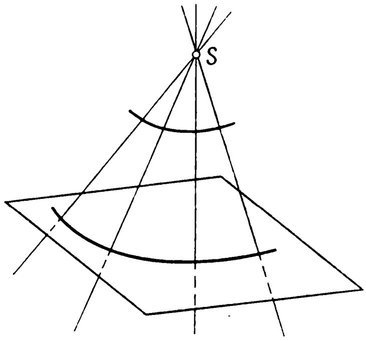

Fig. 312

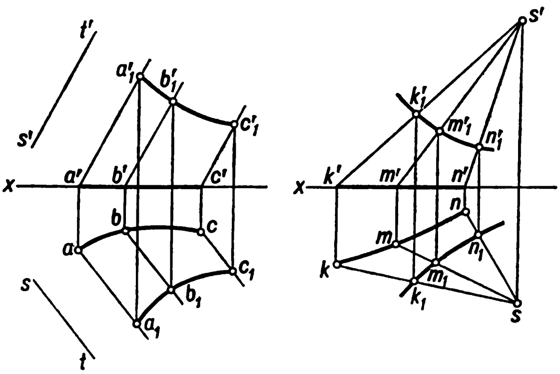

Fig. 313

visible parts in the projections for which purpose continuous and dashed lines are used in Fig. 314.

An analogous construction is given in the right-hand drawing of the same figure for a conical surface. In this case both projections of the element *SB* appear to be boundary: one for the vertical projection, the other for the horizontal projection of the cone.

As it was mentioned at the beginning of this section, points on cylindrical and vertical surfaces can be constructed with the aid of the elements passing through them. In some cases, when formulating the task, it is necessary to indicate whether the required element is visible or invisible*.

Figure 314 (left) shows the construction of the horizontal projection of a point *E* belonging to a cylindrical surface and specified by the vertical

*This information is sometimes given by putting the corresponding projection in brackets. For instance, (e') means that point *E* is located on that part of the surface which is regarded as invisible on the *V* plane.

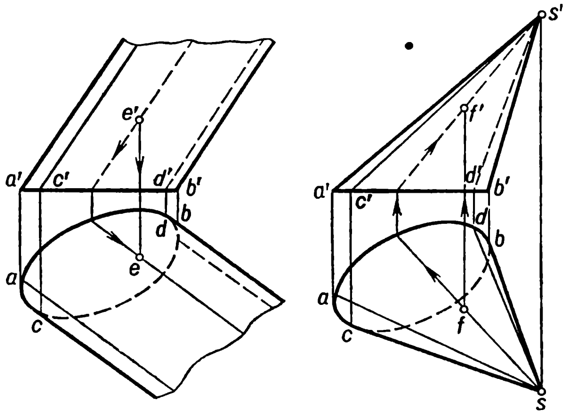

Fig. 314

projection e'; by hypothesis, the point E is invisible on the V plane. The right-hand drawing gives an example of constructing the vertical projection of a point F belonging to a conical surface and specified by the horizontal projection f, on condition that this point is visible on the plane H. In both cases the construction is carried out with the aid of a corresponding element and explained by arrows.

If the generating curve (situated in space or representing the trace of a surface on a projection plane) is replaced by a *polygonal* line inscribed in it, then a cylindrical surface is substituted by *prismatic*, and a conical surface by *pyramidal* (i.e. by the faces of a polyhedral angle). This interconnection between these surfaces will be used in further constructions (for instance, when developing cylindrical and conical surfaces—see Sec. 68).

Cylindrical surfaces differ in the form of normal sections, i.e. in the shape of a curved line obtained as the intersection of this surface with a plane perpendicular to its elements.

Let us single out the cases when a normal section of a cylindrical surface represents a curve of the second order*. Such a cylindrical surface belongs to *second-order surfaces*. Points of any surface of the second order satisfy a second-degree equation in Cartesian coordinates in space. *Any plane cuts such a surface in a second-order curve***. A straight line pierces a second-order surface always at two points.

According to the shape of a normal section a second-order cylinder may be *elliptic* (in a particular case—*circular*), *parabolic*, or *hyperbolic*. The lateral surface of a *right circular cylinder* known from three-dimensional geometry is a second-order surface. A sphere can be inscribed only in a circular cylinder.

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*For second-order curves see Sec. 21.

**For the cases of intersection alongt straight lines see below.

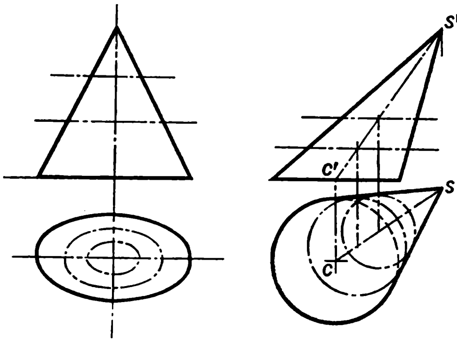

Fig. 315

If the normal section is an indefinite geometric line, then it is a *cylinder of the general form.*

A conical surface intersected along a second-order curve is a surface of the second order (*a cone of the second order*).

Through the vertex of a right circular cone we can pass an infinite number of planes of symmetry of this cone which intersect along a straight line representing the axis of the cone. A sphere can be inscribed in such a cone. The lateral surface of a right circular cone is a surface of the second order.

Of course, the axis of a circular cone may occupy any position with respect to the projection planes which can be reduced to a simplest one (for instance, perpendicular to the H plane).

Figure 315 (left) represents a cone having a system of similar and similarly arranged ellipses* (in the given example they lie in planes parallel to the H plane). Such a cone is called *elliptic.* It goes without saying that it is cut (as any cone of the second order) by planes not passing through its vertex in circles, ellipses, parabolas, and hyperbolas, and each of these curves may be taken for the directrix. Therefore, the term 'elliptic' should not be understood as an indication that just an ellipse should be preferably chosen as the directrix.

An elliptic cone may be regarded as a right circular cone transformed by compressing it in the plane of axial section. For circular sections of such a cone see Sec. 63.

The base of the cone shown in the right-hand drawing of Fig. 315 is also a circle but the projection of its vertex on the plane of the base does not coincide with the centre of the circle. This is an *oblique circular* cone. Intersecting its lateral surface by planes parallel to the plane of the base, we get a number of circles whose centres are located on the straight line passing through the vertex and the centre of the base of the cone (the line SC in Fig. 315).

*Ellipses with proportional and respectively parallel axes (so-called homothetic ellipses).

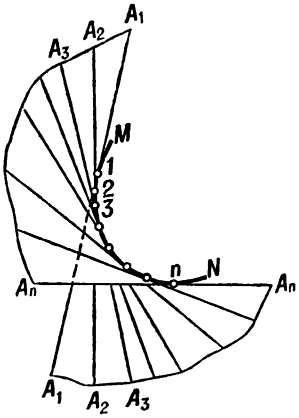

Fig. 316

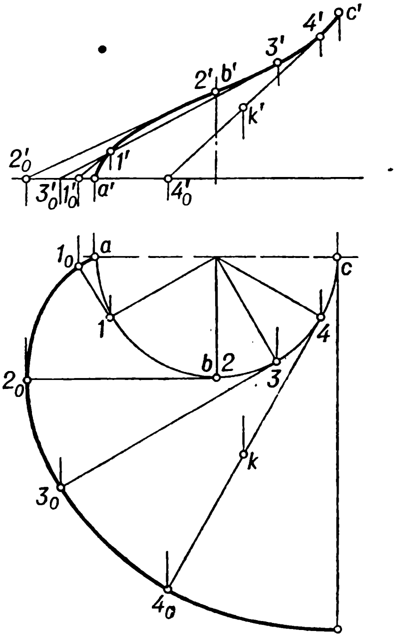

Fig. 317

2. The surface called a *surface with a cuspidal edge** is generated by a rectilinear generatrix performing continuous motion and touching a space curve at all its positions. This space curve serves as the *directrix* for the surface under consideration and is called the *cuspidal edge.*

Such a surface is shown in Fig. 316; its elements A_1A_1, A_2A_2, etc. are tangent to a space curve MN. The cuspidal edge separates the generated surface into two sheets (according to the division of each tangent into two parts at the point of tangency).

Obviously, by specifying the projections of the cuspidal edge, we can specify a surface in the drawing. For instance, taking a cylindrical helix (see Sec. 48) as a cuspidal edge and drawing a number of tangents to it, we thus specify a surface. If the axis of a helix is arranged perpendicular to the H plane, then the surface thus formed will represent a surface of constant slope (with respect to the H plane), since all the tangents to the helix cut the H plane at one and the same angle (see Sec. 48). The drawing of such a surface (of one of its sheet) is given in Fig. 317, where to the arc ABC of a cylindrical helix several tangents are drawn. This is done with the aid of an involute $a1_02_03_04_0$ as a locus of horizontal traces of the tangents (see Fig.

*This surface is also called the *torse*, the latter term meaning also a developable surface.

306). The constructed element of the surface faces the viewer with its convex side. The same drawing shows how the projection k' of a point K belonging to a given surface is constructed from a given projection k: drawing through the point k a tangent to the semi-circle *abc* and using points 4_0 and 4, we find their vertical projections $4'_0$ and $4'$, and thus the projection of the tangent on which the point K is situated. The line of recall drawn from k determines the required projection k'.

If the vertical projection of a point belonging to a given surface is given and it is required to find its horizontal projection, then we have to pass a plane at the level of the given vertical projection of the point to intersect the surface (for the intersection of a surface by a plane see Sec. 55 *et al.*). The required horizontal projection of the point must belong to the horizontal projection of the line yielded by the section. In this case it is advisable to use a horizontal plane which will cut the surface in question along an involute.

A cylindrical and a conical surfaces may be regarded as produced from a surface with a cuspidal edge on condition that the cuspidal edge represents a point: (1) at infinity for a cylindrical surface, and (2) at a finite distance for a conical surface.

If the directrix is a plane curve, then the surface defined by the tangents to such a curve represents a plane.

If a surface with a cuspidal edge is cut by a plane not passing through its element, then we obtain a curve with a cusp (see Item 1) lying on the cuspidal edge. Hence the term 'the cuspidal edge'.

*B. Nondevelopable Ruled Surfaces**

1. Surfaces with a Plane Director. 1.1. *Cylindroids and conoids.* The surface called a *cylindroid* is generated by a moving straight line which in all its positions remains parallel to a given plane (called "the *plane director*") and intersects two curved lines (two *directrices*). If the directrices are plane curves, then, of course, they must lie in different planes.

Figure 318 shows a cylindroid generated by a straight line AD moving along directrices ABC and DEF parallel to the plane director P (which is in the present case a horizontal projecting plane). As is obvious, to construct the drawing, the following should be given: the projections of the directrices and the position of the plane director.

The surface of a *conoid* is generated by a moving straight line which all the time remains parallel to a given plane (called the *plane director*) and intersects two *directrices* one of which is a curve, the other being a straight line. If the curve is a plane one, then it must not lie in one and the same plane with the second directrix which is a straight line.

*Also called 'warped surfaces'.

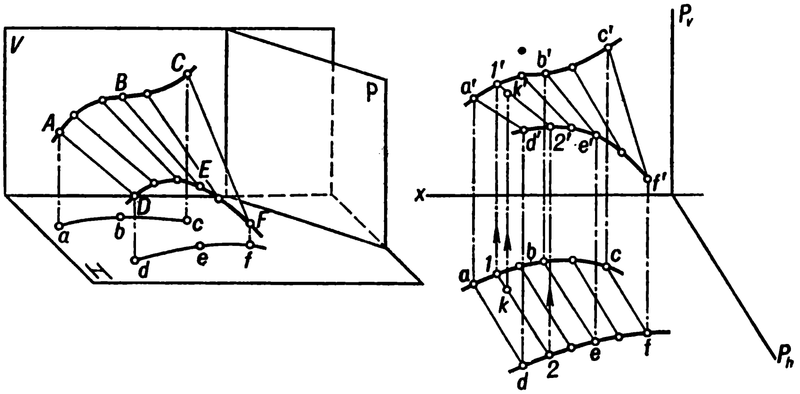

Fig. 318

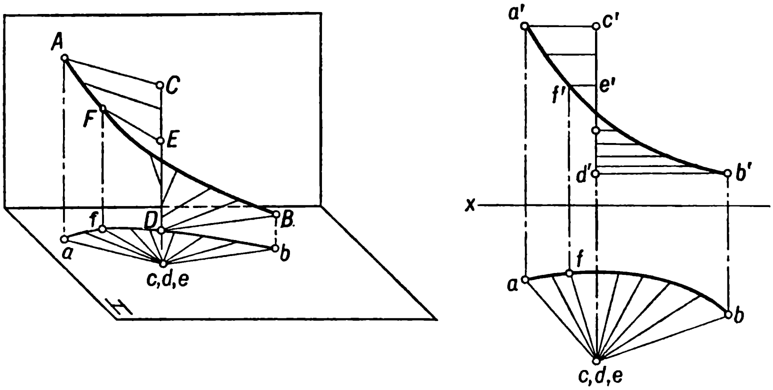

Fig. 319

A conoid is shown in Fig. 319. Here the plane *H* is taken for a plane director, and the rectilinear generatrix intersects both the curve *AFB* and the straight line *CD* which is in this case perpendicular to the *H* plane*.

Any plane parallel to the plane director intersects a cylindroid and conoid along a straight line. Hence, if it is required to construct an element of a cylindroid or a conoid it is necessary to pass a specified plane parallel to the plane director, to find the points of intersection of the directrices of the surface with this plane, and draw a straight line (the required generatrix) through these points. In the particular case shown in Fig. 319, to construct the element of the conoid passing through the point *E* on the straight-line

*Conoids may be exemplified by the surfaces *SACDS* and *SBCDS* represented in Fig. 265 which, together with the triangles *ASB* and *ABC*, bound the solid shown.

directrix, we can do without an auxiliary secant plane, since the vertical projection of the generatrix must be parallel to the x-axis. It is sufficient to draw $e'f'$ parallel to the x-axis, to find the point f (using the point f') and then the horizontal projection ef.

The right-hand drawing of Fig. 318 shows how to find the projection k' of a point K belonging to the cylindroid if the projection k is given. Through k a plane (it is not shown in the drawing) is passed parallel to the plane director P. The intersection yields a straight line with the projections 1-2, $1'2'$ and the projection k' on $1'2'$.

If the vertical projection of a point belonging to a cylindroid is given, and it is required to find its horizontal projection, then proceed as follows: pass a plane cutting the cylindroid and containing the given point. For instance, the cylindroid represented in Fig. 318 should be cut by a horizontal plane at the level of the given vertical projection of the point. Then we construct the horizontal projection of the line of intersection and on it the required horizontal projection of the point.

On a conoid the projections of a point are constructed in a similar way.

1.2. *Hyperbolic paraboloid (a warped plane).* Figure 320 illustrates a surface called the *warped plane* or *hyperbolic paraboloid*, which is also called the *ruled paraboloid.* The surface of a hyperbolic paraboloid is determined by a plane director and two noncoplanar (skew) straight-line directrices. A straight-line generatrix moving along the directrices (and remaining parallel to the plane director) describes the surface of a hyperbolic paraboloid. In Figure 320 the plane director is the H plane, the straight lines AB and CD being the directrices.

The same figure shows the construction of the projection k, using the given vertical projection k' of a point belonging to a warped plane. The construction is simple and is reduced to drawing the vertical projection $m'n'$ of the generatrix at the level of the point k' in accordance with the given plane director.

If the horizontal projection k is given, then, to find the vertical projection k', we have to pass a secant plane so that it passes in space through the point K, i.e. to proceed in the way described for a surface with a cuspidal curve.

It is proved in analytic geometry that a hyperbolic paraboloid can also be obtained as a result of such a motion of the parabola BOB_1 (Fig. 321) during which its axis of symmetry remains parallel to the z-axis, its vertex displaces along the parabola AOA_1, and the plane containing the parabola BOB_1 remains parallel to the plane xOz. A plane parallel to the plane xOy cuts the hyperbolic paraboloid in a hyperbola (if such a plane passes through the vertex O then the hyperbolic paraboloid is intersected along two straight lines passing through the point O). Planes parallel to the planes xOz and yOz intersect the hyperbolic paraboloid in parabolas. Hence, the name of the surface—a "hyperbolic paraboloid".

Figure 322 demonstrates a warped plane generated by a straight-line generatrix AB moving along skew lines AD and BC contained in mutually

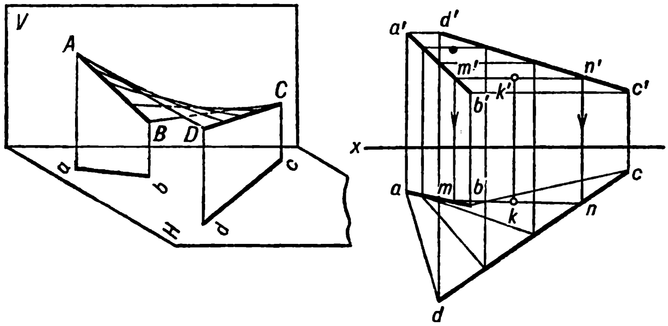

Fig. 320

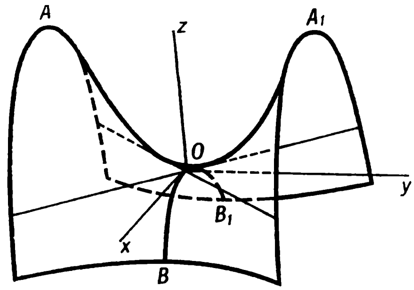

Fig. 321

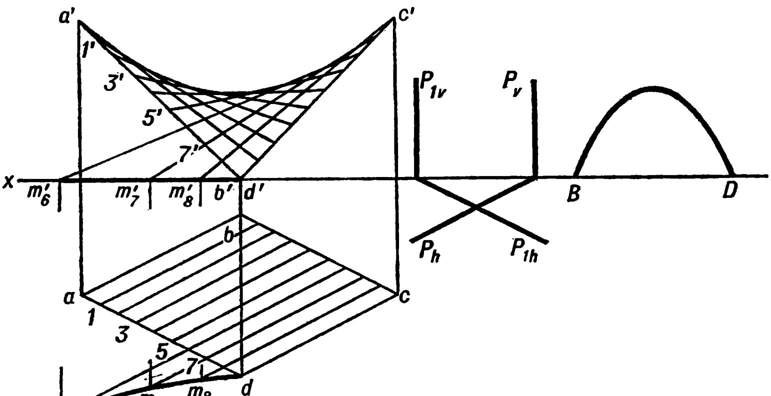

Fig. 322

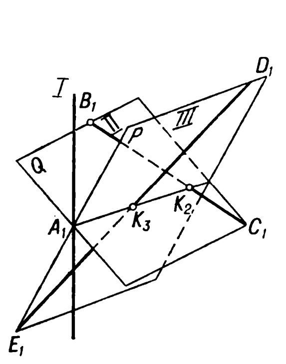

Fig. 323

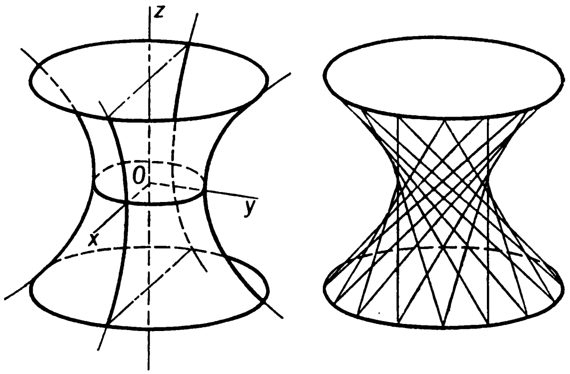

Fig. 324

parallel planes; the plane director is P. Obviously, the same surface is obtained if a straight line AD taken as the generatrix displaces along the directrices AB and CD parallel to the plane P_1. Hence it follows that through any point of the warped plane it is possible to draw two straight lines belonging to this plane.

Figure 322 shows a parabola corresponding to the parabola AOA_1 represented in Fig. 321. Furthermore, the parabola obtained as the curve of intersection of the warped plane with a profile plane passing through the points B and D (parabola BOB_1 in Fig. 321) is constructed in the same way. To construct the hyperbola along which the warped plane is cut by the H plane, we have to find the horizontal traces of the generating elements as it is done in Fig. 322 for some of them.

Thus, *for the above considered surfaces, i.e. for the cylindroid, conoid, and warped plane (hyperbolic paraboloid), the generatrix is a straight line which must simultaneously intersect two directrices and remain all the time parallel to some plane, and the relative positions of the directrices and plane director must be unchanged.*

2. Surfaces with Three Directrices. 2.1. *Hyperboloid of one sheet.* This is the name for a surface generated by a moving straight line which intersects all the time three skew lines *(directrices)**.

If a point A_1 (Fig. 323) is taken on one of the three given skew lines (say, on line I), and planes Q and P are passed through this point and the remaining two lines (lines II and III), then the planes will intersect along a straight line passing through the point A_1 and intersecting line II at point K_2, and line III at point K_3. If all the points of line I are taken as initial points and for each of them such straight lines as $A_1K_2, \ldots,$ are constructed in the above fashion, then they will generate a surface called the *hyperboloid of one sheet.*

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*If all the directrices are parallel to one and the same plane, then a rectilinear generatrix, displacing along these directrices, generates a *warped plane.*

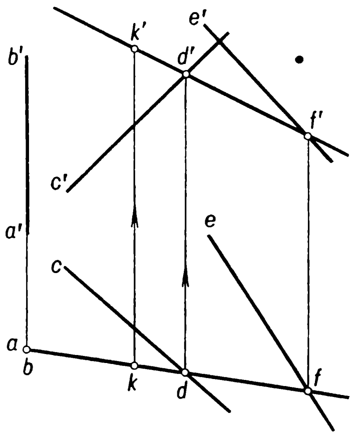

Fig. 325

Practically, we take a number of points on line *I* and construct the corresponding elements. In Figure 323 we could confine ourselves to constructing only one plane, say the plane Q containing line *II*, and find the point of intersection K_3 of line *III* with the plane Q.

In analytical geometry it is proved that the hyperboloid of one sheet can be obtained as a result of motion of a deformable ellipse (Fig. 324, left) whose plane remains parallel to the plane xOy and the end-points of whose axes slide along hyperbolas contained in the planes xOz and yOz. The right-hand picture of Fig. 324 shows a hyperboloid of one sheet with rectilinear generating elements on its surface. If the ellipse is substituted by a *deformable circle*, then both directing hyperbolas will be equal to each other. In this case the obtained surface is called the *hyperboloid of revolution of one sheet* (see Sec. 51).

Through any point on the hyperboloid of one sheet we can draw two straight lines belonging to this surface. This was first mentioned for the hyperbolic paraboloid.

Figure 325 demonstrates a hyperboloid of one sheet specified by three skew lines of arbitrary positions. One of these lines is perpendicular to the H plane—a position which can be always obtained using, for instance, the method of replacing projection planes. The drawing shows the construction of the vertical projection k' of a point K belonging to the hyperboloid of one sheet and specified by its horizontal projection k. Drawing a straight line through the points a and k, we get the horizontal projection of the generatrix, and then, with the aid of the points d and f, we construct the projections d' and f' which determine the vertical projections of this generatrix and the required point k' on it.

If we are given not the horizontal but the vertical projection of a point K belonging to the hyperboloid of one sheet, and none of the directrices is

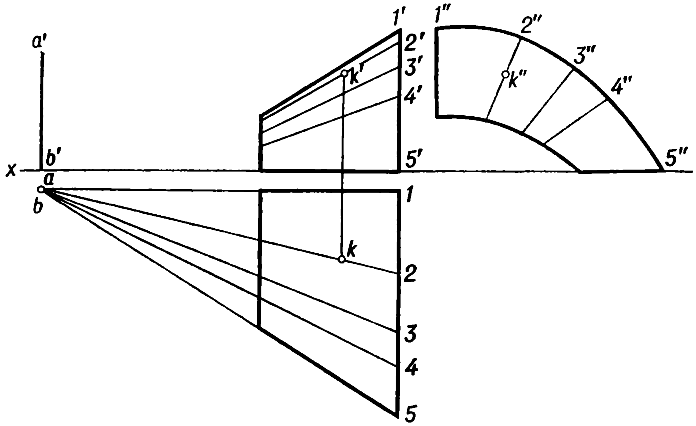

Fig. 326

perpendicular to the V plane, then the hyperboloid should be cut by a plane so that the latter passes through the point K, as mentioned above.

2.2. *Warped cylinder with three directrices.* This surface is generated by a straight line displacing along three directrices of which at least one is a curved line (see Sec. 63).

If the directrices are skew lines, then a hyperboloid of one sheet is obtained (see Item 2.1). We may come across a case when one of the directrices is a plane curve. Then it must not be coplanar with either of the two skew lines (the remaining directrices). If a surface is generated by two curved and one straight lines, then such a cylinder is called the *conusoid.* An example is given in Fig. 326, where it is specified by two curves located in profile planes and a straight line AB perpendicular to the H plane. The horizontal projections of the generatrices pass through the point $a(b)$, their vertical projections intersecting the projections $a'b'$ at different points. To construct the vertical and profile projections of a point K belonging to the conusoid and specified by the horizontal projection k, proceed as follows: draw a straight line through the points a and k to get the horizontal projection of the generating element, construct the remaining projections of this element and the projections k' and k'' on them. If, for instance, the vertical projection k' is given and it is required to find the horizontal projection k, then take advantage of the method described under the heading A.

Warped cylinders with three directrices are widely used in engineering (in designing rowing screws, propellers, automobile bodies, etc.).

Thus, *the generatrix of the hyperboloid of one sheet and a warped cylinder with three directrices is a straight line which, while moving, simultaneously intersects three fixed directrices.*

C. Double-curved Surfaces

1. Double-curved Surfaces of the Second Order. Considered above were *ruled* surfaces of the second order: the cylinder, cone, hyperbolic paraboloid and hyperboloid of one sheet. Now we are going to study the remaining surfaces of the second order which are *double-curved* surfaces: the ellipsoid, elliptic paraboloid, and hyperboloid of two sheets.

1.1. *The ellipsoid.* This surface is generated by a moving variable ellipse $ACBD$ (Fig. 327) whose plane remains parallel to the plane xOy, and the end-points of whose axes slide along the ellipses $AEBF$ and $CEDF$. If the diameters AB, CD, and EF of this ellipsoid are of different lengths, then it is said to be *triaxial*, if two of them are equal to each other but not equal to the third one, then we obtain a *contracted* or *prolate* ellipsoid of revolution (see Sec. 51). And, finally, if $AB = CD = EF$, then we get a *sphere.* Any plane cuts the ellipsoid in an ellipse, in particular cases—in a circle.

1.2. *The elliptic paraboloid.* An elliptic paraboloid can be obtained as a result of displacement of a variable ellipse $ABCD$ (Fig. 328) whose plane remains parallel to the plane xOy and the end-points of whose axes slide along parabolas AOB and COD. Its intersections with planes are only ellipses (in some particular cases—circles) and parabolas, the latter being obtained when secant planes are parallel to its axis. If the ellipse $ABCD$ is substituted by a variable circle, then both parabolas (AOB and DOC) will be equal to each other. In this case the surface obtained will be called a *circular paraboloid* or *paraboloid of revolution* (see Sec. 51).

1.3. *The hyperboloid of two sheets.* The surface of this hyperboloid consists of two sheets (Fig. 329) spreading to infinity. Either of the sheets can be obtained as a result of displacement of a variable ellipse ($A_1C_1B_1D_1$ or $A_2C_2B_2D_2$) whose plane remains perpendicular to the axis O_1O_2 of the surface, and the end-points of whose axes slide along two hyperbolas. If the ellipse is substituted by a variable circle, then both hyperbolas will be equal to each other ($A_1O_1B_1 = C_1O_1D_1$). In this case the surface is called the *hyperboloid of revolution of two sheets* (see Sec. 51).

Sections by various planes are ellipses (in particular cases—circles), hyperbolas, and parabolas.

2. Cyclic Second-order Surfaces. A *cyclic* surface is generated by a circle of a variable radius whose centre moves along a curve. Let us consider the case when the plane of the generating circle remains perpendicular to the given curved-line directrix along which the centre of the circle displaces. The surface thus generated is said to be a *canal* surface. It may also be thought of as an enveloping surface for a family of variable spheres whose centres are situated on a directing curve. The radius of a generating circle or a generating sphere may be constant. The surface generated by such circle moving along a directing curve or by enveloping all consecutive positions of the generating sphere with the centre moving in the same manner is called a *tubular* surface. In engineering it may be exemplified by equalizers in pipe-lines.

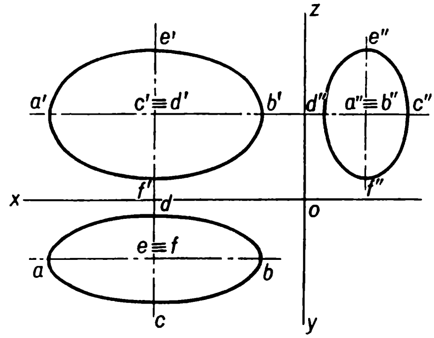

Fig. 327

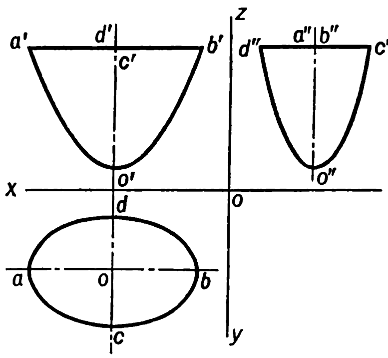

Fig. 328

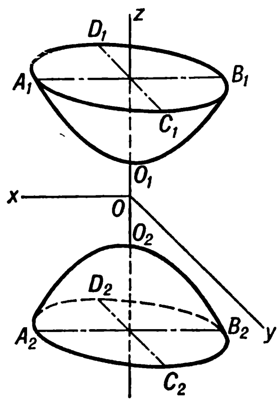

Fig. 329

A cylindrical helix may be used as a directrix for generating a tubular surface. In this case we have a *tubular helical surface.* An example is given in Fig. 349: the surface of a wire of a circular cross-section coiled on a pipe. Another example is the surface of a coiled cylindrical spring with a circular cross-section of coils.

Cyclic surfaces of various shapes are widely used in gas pipings, hydroturbines, centrifugal pumps, etc.

If a straight line but not a curve is taken as a directrix, then a canal surface turns into a surface of revolution (see the next section), into a conical surface in particular, while a tubular surface turns into the surface of a cylinder of revolution under the same condition.

D. *Surfaces Specified by a Network*

These are surfaces specified by a number of lines belonging to such a surface. In a particular case we can imagine one group of certain plane curves each contained in parallel planes, and another group of lines intersecting the lines belonging to the first group thus forming a network of a surface.

A surface specified by a framework cannot be considered as quite definite, since there may be surfaces with one and the same network but still differ from one another.

Network surfaces may be exemplified by the surfaces of hulls, airplanes, and automobiles.

E. *Graphical Surfaces*

Any surface may be specified graphically*. For many surfaces the generating and directing lines (generatrices and directrices) are geometrically defined, and the surfaces are generated according to certain laws. On the other hand, there are surfaces for which such conditions are not stipulated. In the latter case surfaces are specified only graphically with the aid of a number of lines which must (according to designer's project) belong to such a surface or be detected on an existing surface. Such surfaces are called *graphical.*

This group of surfaces comprise also a so-called *topographical* surface, i.e. Earth's surface from the point of view of its representation. Its relief is usually represented by contour lines obtained as sections of this surface by horizontal planes.

QUESTIONS TO SECS. 49-50

1. What is the surface?
2. How is a kinematic surface generated?
3. What is the generatrix (generating line)?
4. What is the difference between a ruled and a double-curved surfaces?
5. Can a surface be generated not by a line but by a surface?
6. What is the directrix?
7. What surfaces belong to double-curved (nondevelopable) surfaces?
8. What is meant by "to specify a surface in the drawing"?
9. How are a cylindrical and a conical surfaces, a surface with a cuspidal edge generated and how are they specified in drawings?
10. What is a second-order surface and in what lines is such a surface cut by planes?

*That is by drawing; from the Greek 'grapho' meaning 'I write'.

11. How are cylindrical surfaces distinguished?

12. What cone is called elliptic? What is an oblique circular cone?

13. How is a surface with a cuspidal edge specified in the drawing? How else is this surface called?

14. How are surfaces with a plane director generated?

15. What lines are the directrices of a cylindroid and a conoid?

16. How is a warped plane (a hyperbolic paraboloid) generated?

17. In what lines is a hyperbolic paraboloid cut by planes parallel to the planes of coordinates?

18. How many straight lines belonging to a hyperbolic paraboloid can be drawn at any of its points?

19. How is a hyperboloid of one sheet generated?

20. How many straight lines belonging to a hyperboloid of one sheet can be drawn at any of its points?

21. How is a warped cylinder with three directrices generated?

22. In what case is a warped cylinder with three directrices called the conusoid?

23. List the ruled and double-curved surfaces of the second order.

24. May a sphere be interpreted as an ellipsoid and in what case?

25. In what curved lines is an ellipsoid cut by planes?

26. What is an elliptic paraboloid?

27. What are the plane sections of an ellipsoid?

28. What are the plane sections of a hyperboloid of two sheets?

29. What surfaces are called 'cyclic'?

Sec. 51. Surfaces of Revolution

A *surface of revolution* is generated by the revolution of a curved-line or straight-line generatrix about a fixed straight line called the *axis of the surface.*

A surface of revolution can be specified by the generatrix and the position of the axis. Such a surface is shown in Fig. 330 where the generatrix is a curved line *ABC*, and the axis is a straight line OO_1 which is coplanar with the curve *ABC*. Each point of the generatrix describes a circle. Hence, a plane normal to the axis of the surface of revolution cuts this surface in circles. Such circles are called *parallels.* The greatest parallel is called the *equator*, the smallest parallel the *throat* of a surface*.

A plane passing through the axis of a surface of revolution is termed a *meridian plane.* The line along which a meridian plane intersects a surface of revolution is called the *meridian of the surface.*

*A surface of revolution may have several equators and throats.

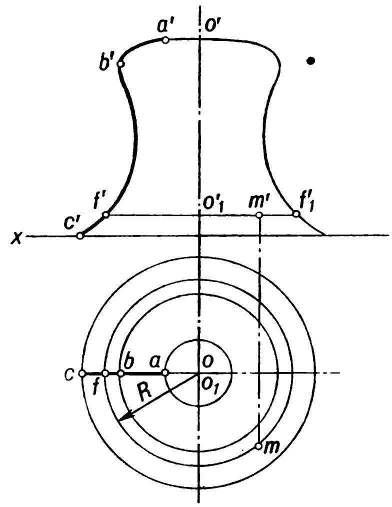

Fig. 330

The point of intersection of a meridian of a surface with its axis may be called the *vertex* of this surface provided they do not intersect at right angles.

If the axis of a surface of revolution is parallel to the V plane, then the meridian contained in the plane parallel to the V plane is termed the *principal meridian*. In such a position the principal meridian is projected on the V plane without twisting. If the axis of a surface of revolution is perpendicular to the H plane, then the horizontal projection of the surface is a circle.

In drawings, the axis of a surface of revolution is ordinarily made perpendicular to one of the projection planes (H, V, or W).

Some surfaces of revolution represent particular cases of the surfaces considered in the preceding section. These are: (1) a cylinder of revolution, (2) a cone of revolution, (3) a hyperboloid of revolution of one sheet, (4) an ellipsoid of revolution, (5) a paraboloid of revolution, and (6) a hyperboloid of revolution of two sheets.

For the *cylinder* and *cone of revolution* meridians are straight lines parallel to the axis and equidistant from it in the first case, and intersecting the axis at one and the same point at one and the same angle to the axis in the second case. Since the cylinder and cone of revolution are surfaces spreading to infinity in the direction of their generatrices, their representations are usually bounded by some lines, for instance, by the traces of these surfaces on the projection planes or by one of the parallels. A *right circular cylinder* and a *right circular cone* known from solid geometry are bounded by a surface of revolution and planes perpendicular to its axis. The meridians of such a cylinder are rectangles, and those of the cone are triangles.

For the *hyperboloid of revolution* the meridian is a hyperbola. If the hyperbola is revolved about its real axis, then a hyperboloid of revolution of two

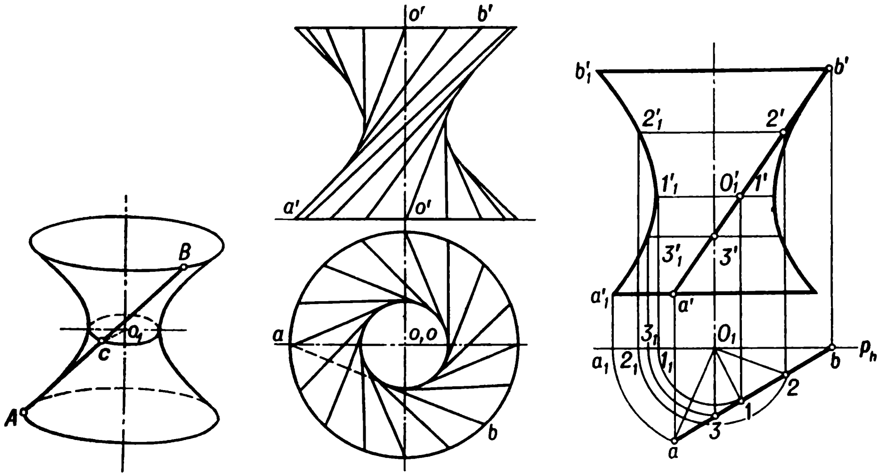

Fig. 331

sheets is generated; if it is revolved about its imaginary axis, then we have a hyperboloid of revolution of one sheet.

A *hyperboloid of revolution of one sheet* can also be generated by the revolution of a straight line if the *generatrix and the axis of revolution are skew lines.* Figure 331 shows a hyperboloid of revolution of one sheet generated by revolving a straight line AB about the indicated axis and bounded by two parallels. The circle described from centre O_1 is the throat of the surface.

On a hyperboloid of revolution of one sheet it is possible to draw rectilinear elements in two directions: in the way shown in Fig. 331 and inclined to the other side at the same angle to the axis.

Besides straight lines, this surface can also be cut in hyperbolas (by planes passing through its axis) and circles (by planes perpendicular to the axis).

The right-hand drawing of Fig. 331 illustrates the construction of the vertical projection of a hyperboloid of revolution of one sheet by its axis and generatrix. First of all the radius of the throat is found. To this end a perpendicular $o_1 1$ to the horizontal projection of the generating element AB is drawn. This determines the horizontal projection of a common perpendicular to the axis and to the generatrix. The true length of the line segment represented by the projections $o'_1 1'$ and $o_1 1$ is equal to the radius of the throat of the surface. Then, using the method of revolution, the points represented by the projections $2'$, 2; $3'$, 3; a', a are brought to the plane P which is parallel to the V plane. This enables us to draw the outline of the vertical projection of the hyperboloid. Its horizontal projection will be represented by three concentric circles.

The meridian of the paraboloid of revolution is a parabola whose axis serves as the axis of the surface.

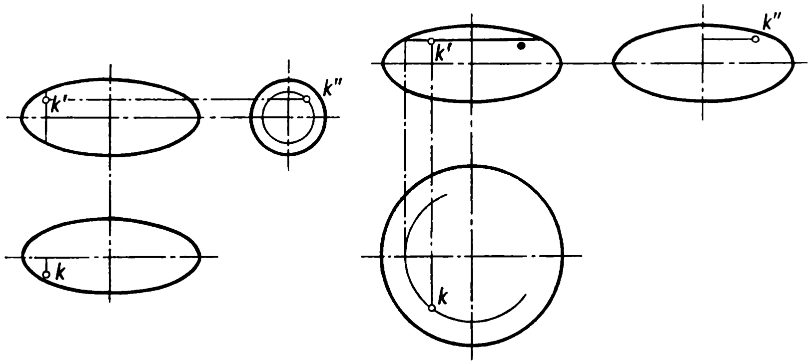

Fig. 332

The meridian for the ellipsoid of revolution is an ellipse. The surface can be generated by revolving an ellipse about its major axis (a *prolate ellipsoid of revolution* is shown in Fig. 332, left) or about its minor axis (an *oblate ellipsoid of revolution* is illustrated in Fig. 332, right). The ellipsoid of revolution is a bounded surface, and therefore it can be represented completely. The same refers to the sphere for which the equator and meridians are congruent circles.

We emphasize once again that the cylinder, cone, and hyperboloid of one sheet are ruled surfaces, i.e. they can be generated by revolving a straight line*. In contrast to them, the ellipsoid, paraboloid, and hyperboloid of two sheets are generated by revolving not a straight line but an ellipse, parabola, and hyperbola, respectively, the axis of revolution being chosen so that the generating curve is arranged symmetrically with respect to this axis. The same may be said about the hyperboloid of revolution of one sheet if it is generated by revolving a hyperbola about its imaginary axis.

Since the axis of revolution is chosen to coincide with the axis of symmetry of an ellipse, parabola or hyperbola, the ellipse and hyperbola form two surfaces each, since either of them has two axes of symmetry, whereas the parabola generates only one surface, for it has one axis of symmetry. Hence, each of the considered surfaces is generated by revolving the generatrix only in a single way, the sphere being the only exception. The sphere which may be considered as an ellipsoid with equal axes of the generating ellipse (circle) can be generated by rotating a circle in many fashions, since the circle is symmetrical about any of its diameters.

When revolving a circle (or its arc) about an axis lying in the plane of this circle but not passing through its centre, we get a surface called the *torus*.

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*The regularity in the location of rectilinear generating elements of the hyperboloid of revolution of one sheet is used in a design known under the name "Shukhov's tower". V. G. Shukhov (1853-1939) was an outstanding Russian engineer. "Shukhov's tower" is applied in constructing radio masts, water-towers and other projects.

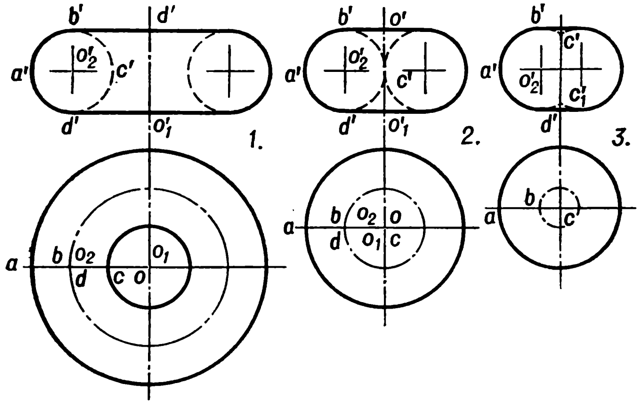

Fig. 333

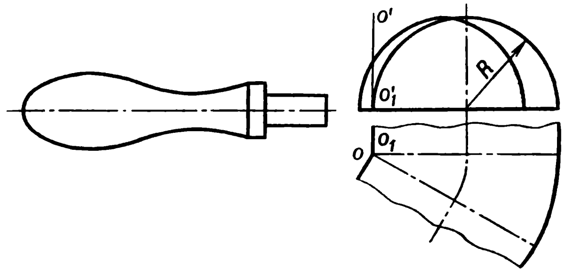

Fig. 334

The solid bounded by this surface is also called the torus (or the anchor ring).

Shown in Figure 333 are different shapes of the torus: (1) *an open torus* or *annular torus* (its axis does not intersect the generating circle), (2) *a closed torus* (its axis is tangent to the generating circle), and (3) a *self-intersecting torus* (its axis intersects the generating circle). All of them are represented in the simplest position: the axis of the torus is perpendicular to the H plane.

An open and a closed tori are generated by a circle, while a self-intersecting torus by a circular arc. Spheres can be inscribed in an open and closed tori. The torus may be regarded as a surface enveloping congruent spheres whose centres are located on a circle.

The torus has two systems of circular sections: (1) by planes perpendicular to its axis, and (2) by planes passing through its axis.

The surface called the torus is quite frequently used in machine-building and architecture. Figure 334 shows a mechanical part whose surface of revolution comprises a self-intersecting and an open tori. The right-hand drawing of the same figure represents schematically a curved surface which is a passage between two cylindrical arches and has the shape of a closed torus with the axis OO_1. •

We finish our review of surfaces of revolution with the *catenoid**. This surface is generated by a complete revolution of a *catenary*** about its horizontal axis lying in the same plane.

The position of a point on a surface of revolution is determined by a circle passing through this point on the surface of revolution.

But this does not exclude the possibility of using rectilinear generators when dealing with ruled surfaces of revolution, as it was shown in Fig. 314 for cylinders and cones of the general form.

Figure 330 demonstrates the use of a parallel for constructing the projection of a point belonging to a given surface of revolution. If the vertical projection m' is given, then we draw the vertical projection $f'f_1'$ of the parallel, and describe a circle of radius $R = o_1'f'$ which is the horizontal projection of the parallel, and on it we find the horizontal projection m. If we are given the horizontal projection m, then a circle of radius $R = om$ should be described, point f' found with the aid of f, and the vertical projection of the parallel $f'f_1'$ drawn, on which the vertical projection m' must be situated. Figure 332 shows the construction of the projections of a point K belonging to an ellipsoid of revolution, and Fig. 335 of a point M belonging to the surface of an annular ring.

How the projections of a point on a sphere are found is demonstrated in the right-hand drawing of Fig. 335. The vertical projection a' of point A is constructed from its horizontal projection a; the horizontal projection b of point B is found using its vertical projection b'. The point B satisfies an additional condition consisting in that it should be invisible on the vertical projection.

A point C is given on the equator and therefore its horizontal projection c is found on the horizontal projection of the sphere, i.e. on the horizontal projection of the equator. Points K and M are situated on the principal meridian; they belong to the parallels on which the points A and B are located. Point D is also on the principal meridian, its horizontal projection being invisible.

Let us now consider an example of constructing the projections of points belonging to a surface of revolution. Suppose it is required to bring a point A to a given surface of revolution by revolving it about a given axis MN (Fig. 336, *a*). Since in this case the axis of the surface of revolution and the axis of the revolution of the point A are perpendicular to the H plane, the circle of revolution of the point A is projected on H without distortion as also the parallel of the surface of revolution obtained as the intersection of

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*Catena (Latin)—chain.

**Catenary is the plane curve in which a chain of uniform density will hang when suspended from two points.

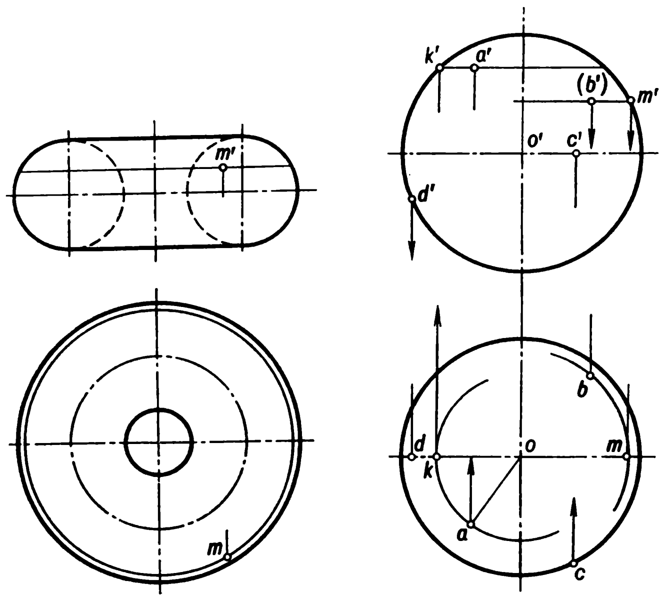

Fig. 335

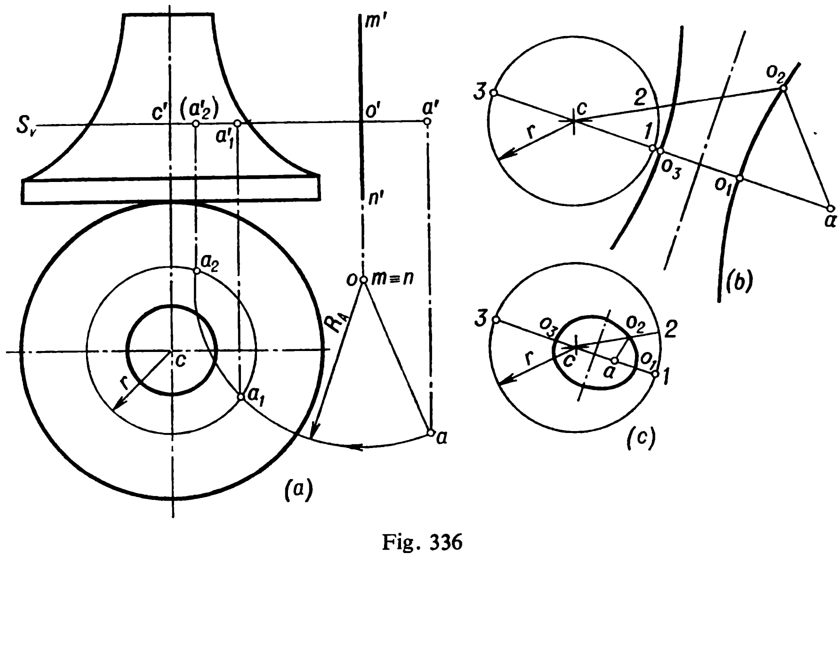

Fig. 336

this surface and the plane of revolution of the point A. This plane also contains the centre of revolution of the point A (point O which is the point of intersection of the axis of revolution MN with the plane of revolution S). The rest of the construction is obvious from the drawing. In the position A_2 on the surface the point will become invisible on the V plane.

Suppose we consider the choice of the axis of revolution required to bring a given point A to a given surface of revolution. In Sec. 38 we came across an analogous situation, but there we had to choose an axis to introduce a point into a plane by revolving the point about this axis. We established there that there was a zone within which it was impossible to take axes, since, when revolved about such an axis, the point would not touch the plane. This zone was determined by a parabolic cylinder, and a parabola appeared when considering the relative positions of the point revolved and the straight line on which this point had to appear after getting into contact with the plane.

Obviously, now the question is to be answered when considering the relative positions of the point A and a circle (parallel) on the surface of a solid of revolution.

It follows from Fig. 336, *a* that the projection o of the centre of revolution must be located so that R_A is not less than the distance of the point o from the nearest point on the projection of a circle of radius r. But if we take a point o equidistant from a and from the projection of this circle (for example, o_1 or o_2—see Fig. 336, *b*), then the axis may be erected at this point, since the circle of revolution of the point A will touch the circle of radius r, i.e. the point A will turn out to be in contact with the surface of revolution.

Where do all the points equidistant from the point a and from the circle of radius r lie in the drawing? They are located on a hyperbola (Fig. 336, *b*) for which the point a serves as one of the foci, and for instance point o_1, at which the line segment *a1* is bisected, as one of the vertices. To obtain the second vertex of the hyperbola (point o_3), we bisect the line segment *a3*. The second focus will be situated at point c, i.e. in the centre of the circle yielded by the intersection of the surface of the solid of revolution and the plane S (Fig. 336, *a*).

It follows from the above reasoning that any of the points on both branches of the hyperbola and between them may be chosen as the horizontal projection of the axis of revolution.

We may come across a case when the point is located inside a surface of revolution. Hence, passing through the point a plane of rotation, we get the horizontal projection a inside the projection of the circle of radius r in which the plane of revolution of the point A cuts the surface of revolution (Fig. 336, *c*). Obviously, this time again, R_A must not be less than the distance of the point o (i.e. the projection of the axis) from the nearest point of the projection of the circle of radius r. The limiting positions of the axes will now be located as the points of an ellipse with the foci at points a and c with the major axis on a straight line *1-3*, and with the vertices at points

o_1 and o_3. The projection of the axis should not be taken inside this ellipse, since such an axis will not enable us to bring the point A onto the surface of revolution.

Thus the question how to choose the axis of revolution so that, revolving a point about it, to bring this point onto a plane or a surface of revolution whose axis is parallel to the axis of revolution, has led us to an ellipse (Fig. 336, *c*), parabola (Fig. 244), and hyperbola (Fig. 336, *b*) as the loci of the centres of revolution.

When solving various problems *certain surfaces are used as geometrical loci* of points and lines meeting definite conditions. For instance, given a plane P and a point K outside this plane, determine how the points situated at a given distance r from the point K will be located in the plane P (r is greater than the distance of the point K from the plane P). In this case the solution calls for the use of a sphere as the locus of points situated at a distance r from the point K. The plane P will cut this sphere in a circle which yields the solution for our problem*.

If it is required to construct in the plane P points situated at a distance r not from a point but from a certain straight line AB not lying in the plane P, then the locus of such points in space will be the surface of a cylinder of revolution with the axis AB and radius r, and the required points in the plane P will be obtained on the line of intersection of this cylinder with the plane P.

Figure 368 (right) and Figure 401 give examples of using conical surfaces of revolution as the loci of straight lines passing through a given point.

If we are interested in the points equidistant from a given plane Q and a given point M, then a paraboloid of revolution with the focus of the parabola at the point M should be used as the locus of such points in space.

Of course, the use of various surfaces as geometrical loci is not exhausted by the above examples.

QUESTIONS TO SEC. 51

1. What is called the surface of revolution?

2. How can we specify a surface of revolution?

3. How do we define parallels and meridians on a surface of revolution? How are the equator, throat, and the principal meridian defined?

4. Which axis of a hyperbola serves as the axis of revolution for generating a hyperboloid of revolution of (1) one sheet, (2) two sheets.

5. Can a hyperboloid of revolution of one sheet be generated by a straight line?

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*The reader is invited to make the drawing and to solve this and subsequent problems.

6. What surfaces of revolution are ruled surfaces?
7. How is the torus generated?
8. When is the torus called an annular ring?
9. How many systems of circular sections has the torus?
10. How is the position of a point determined on a surface of revolution?

Sec. 52. Helical Surfaces and Screws

Figure 337 represents one turn of a *helical surface* generated by a moving line segment AB. The straight line defined by the given segment intersects the axis in all positions at one and the same angle (in Fig. 337 this angle is equal to 60°). The displacement of the end-points of the line segment along the axis is proportional to the angular displacement of the line segment.

The points A and B, like all the other points of the segment AB, generate cylindrical helices, and, hence, to get a more exact representation of the outline of the helical surface on the V plane we have to construct as many as possible projections of helices generated by different points of the segment AB, and then to draw the curves enveloping these projections. Practically, instead of this cumbersome construction, straight lines are usually drawn simultaneously tangent to the projections of the helices (see Fig. 345).

If the generatrix is inclined with respect to the cylinder axis at an angle other than 90° (for instance, 60° as in Fig. 337), then *the helical surface is called oblique.* If this angle is equal to 90°, then a *right helical surface* is formed (see Fig. 338).

According to the way of generation the surface shown in Fig. 338 is a *conoid.* Indeed, the generatrix is a straight line which in all its positions remains parallel to a certain plane (in the present case it is perpendicular to the cylinder axis). The generatrix intersects two directrices—a curve and a straight line (the cylinder axis). Since the curved directrix represents a helix, *such conoid is called helical*, or a *right helicoid.*

In Figure 338 a helical conoid is shown together with a circular cylinder. Since they are coaxial, on the surface of the cylinder a cylindrical helix is formed whose lead is equal to the lead of the directing helix. The surface bounded by these helices is called an *annular helical conoid.*

An oblique helical surface represented in Fig. 337 is also called an oblique helicoid. The leading feature of this surface is that a rectilinear generatrix in all its positions intersects two directrices: a cylindrical helix and a straight line (the axis of the surface), and the generatrix intersects the axis at a constant angle which is not equal to 90°. In all its positions the generatrix is parallel to the generating elements of a *cone of revolution* whose axis coincides with the axis of the helix (Fig. 339, left). If, for instance, it is required to obtain the vertical projection of the generating element of an oblique helicoid passing through a point C, then it is advisable to begin with drawing the horizontal projection of this element, i.e. to draw the radius sc, to find the

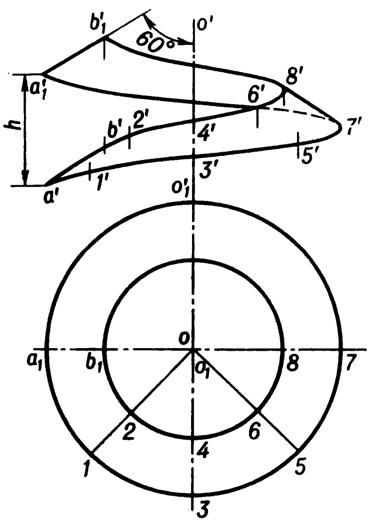

Fig. 337

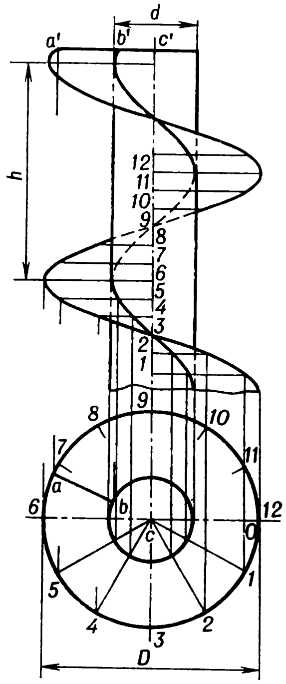

Fig. 338

point c_1' with the aid of the point c_1 and the vertical projection of the element SC_1 of the cone, and then to draw $c'd'$ parallel to $s'c_1'$.

The right-hand drawing of Fig. 339 represents a helix generated by a moving line segment tangent to the surface of a cylinder. The construction is again reduced to finding the projections of helices formed by two points: the end-point A of the line segment and the point of tangency B. The segment may be directed either at a right angle (as it is taken in Fig. 339), or at an acute angle with respect to the axis.

The surface shown in Fig. 339 (right) is a cylindroid (see Fig. 50, *b*). Indeed, in all its positions the generatrix remains parallel to a plane and slides along two directrices which are space curves; the plane director is perpendicular to the cylinder axis; the generatrix touches the surface of the cylinder (the points of tangency form a cylindrical helix), and at the same time intersects the directing helix which is coaxial with the cylinder. The surface represented in Fig. 339 (right) is termed the *helical* (or screw) *cylindroid.* If the generatrix of such surface forming skew lines with the cylinder axis, makes with this axis an angle not equal to 90°, then the obtained surface does not belong to cylindroids; it is named an *oblique annular helicoid.*

The above considered helical (or screw) surfaces belong to double-

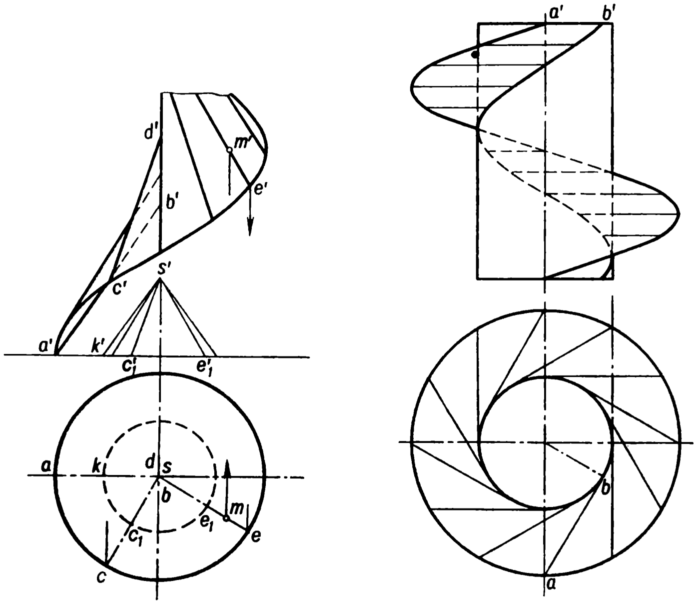

Fig. 339

curved surfaces. But there is a screw surface which is regarded as a developable, namely, a surface with a cuspidal edge, the latter being a cylindrical helix (see Fig. 317). This screw surface is called an *open helicoid.*

Figure 340 shows the surface of an oblique helicoid intersected with a plane T which is perpendicular to the axis of this surface. The curve obtained in the section is represented on the H plane without distortion, since the plane T is parallel to H. This curve is a spiral of Archimedes.

The construction of this curve is reduced to the following: on dividing the angle $a_0c_0c_6$ (180°) into several (in our case into six) equal parts, we divide the line segment c_0c_6 into the same number of equal parts. Then on the radius c_0a_1 from the point c_0 we lay off $c_0c_1 = \frac{c_0c_6}{6}$, on the radius c_0a_2 we lay off $c_0c_2 = 2c_0c_1$, etc.

Now we shall dwell on the construction of points belonging to a right and an oblique helical surfaces. For a right helical surface this is shown in Fig. 338. Let the point A belonging to the surface is specified by its horizontal projection a. To find the vertical projection a', we have to draw the horizontal projection of the element on which the point A must be situated, i.e. to draw the radius cb through the projection a. Using the point b, we then find the point b' and draw the vertical projection of this element coinciding with the straight line $c'b'$. On this line we find the projection a'*.

*If the helical surface is opaque, the point A is invisible with respect to the V plane.

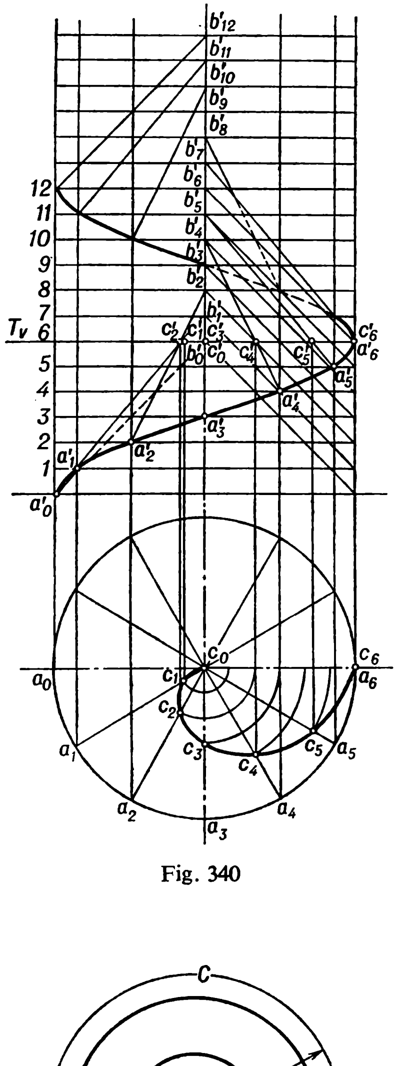

Fig. 340

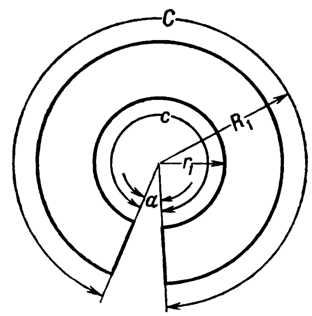

Fig. 341

If the projection a' is given and it is required to find a, then, through a' we first draw a straight line perpendicular to the axis of the helix to intersect the projection of the helix at point b'. With the aid of this point, we find the point b, and on the radius cb—the point a.

The accuracy of construction here depends on the exactness of drawing the sinusoid (the vertical projection of the helix), since the point b' is located on it.

In the case of an oblique helical surface (Fig. 339, left), if the horizontal projection m is given and it is required to find the vertical projection m', then we draw the radius se through the point m, find the points e' and e'_1 with the aid of the points e and e_1, draw the horizontal projection $s'e'_1$ of the element of the cone and parallel to it through the point e' the vertical projection of the generating element of the helical surface. On this projection we obtain the projection m'.

If the vertical projection m' is given and it is required to find the horizontal projection m, then we had to construct the curve (spiral of Archimedes) of intersection of the oblique helical surface and the plane passed at the level of the point m' perpendicular to the axis of the surface, and find the point m on the spiral.

The helical surfaces shown in Figs. 337-340 cannot be developed, or rolled out, on a plane without stretching or shrinking. For a right helical surface represented in Fig. 338, we can get an *approximate* development of each of its separate turns as it is shown in Fig. 341. The development of one turn may be represented (approximately) as a part of an annulus.

To construct this part of the annulus, we have to find the magnitudes of the radii R_1 and r_1 and the angle α. If we denote the lead of the helical surface (Fig. 338) by h, and the outer and inner diameters of the cylinder by D and d, respectively, then, according to the formula indicated in Sec. 48, the lengths of the corresponding segments of the helices will be expressed as follows:

$$C = \sqrt{\pi^2 D^2 + h^2} \quad \text{and} \quad c = \sqrt{\pi^2 d^2 + h^2}.$$

Since in this case the helices are developed into concentric arcs with one and the same central angle, we have $c : C = r_1 : R_1$, and, consequently,

$$r_1 = \frac{c}{C} R_1.$$

Designating by a the width of the helix, i.e. the difference $R_1 - r_1 = \frac{D-d}{2}$, we get $R_1 = r_1 + a$, whence $r_1 = \frac{c}{C} r_1 + \frac{ac}{C}$, or $r_1 = \frac{ac}{C-c}$. Hence it follows that the angle α can be determined by the formula

$$\alpha = \frac{2\pi R_1 - C}{2\pi R_1} \cdot 360°.$$

Putting $D = 100$ mm, $d = 60$ mm, $h = 50$ mm, we find: $a = 20$ mm, $C \approx 318$ mm, $c \approx 195$ mm, $r_1 \approx 32$ mm, $R_1 \approx 52$ mm, and $\alpha \approx 10°$.

We describe two concentric circles of radii $R_1 = 52$ mm and $r_1 = 32$ mm, construct the central angle $\alpha = 10°$, and thus single out the portion of the ring representing (approximately) the development of one turn of the helical surface.

Having several turns developed in the above way, we can combine each turn with a cylindrical rod of diameter d (as is shown in Fig. 343), and fasten one to another the turns wound on the rod.

Like a helix generated by a point in helical motion, and a helical surface generated by a line segment performing helical motion, a helical body can be obtained if a plane figure (for instance, a square, a triangle, or a trapezoid) is forced to move on the cylindrical surface so that the vertices of this figure displace along helices, and the plane containing the figure constantly passes through the axis of the cylinder. In this case a thread profile is formed bounded by helical and cylindrical surfaces. The construction of such a thread profile is reduced to constructing some helices whose number is equal to the number of vertices which has the chosen plane figure.

The construction of a thread profile generated by a moving square is illustrated in the left-hand drawing of Fig. 342. The square keeps contact (with one of its sides) with the cylindrical surface, its vertices moving along helices.

When threading a cylindrical rod the thread profile is formed by removing a portion of material by means of a cutting tool.

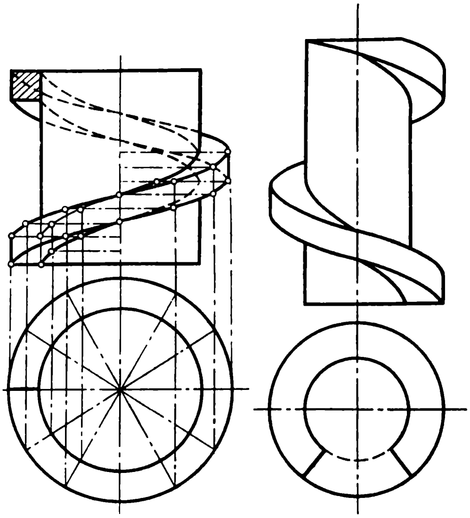

Fig. 342

The thread profile thus obtained is bounded by two right helical surfaces and two cylindrical surfaces (outer and inner), the latter touching the surface of the cylinder. *The combination of a cylinder and a thread profile on it is called the screw.* Represented in the left-hand drawing of Fig. 342 is a *screw with a right-hand thread*, i.e. on the front (visible) side of the cylinder the thread profile rises in the direction from left to right. If it rises in the direction from right to left (see Fig. 342, right), then we have a screw with a left-hand thread (cf. Sec. 48).

Figure 343 shows a thread profile generated by a moving rectangle which contacts the cylinder with its smaller side. Screws of such shape are used in screw conveyors*.

The same figure demonstrates the construction of the vertical projection a' of a point A located on the helical surface and specified by its horizontal projection a. The construction is similar to that in Fig. 338 but this time it is shown how to avoid inaccuracies in drawing the sinusoid. To this end we find the line segment l which determines the displacement of point 1 along the axis of the screw when the generatrix is rotated from the initial position to the position $C1$ (i.e. through an angle $oc1$). Then we take the proportion $x : h = \urcorner\, oc1 : 360°$, whence x is determined thus yielding the magnitude of l. The further construction is obvious from the drawing.

The screws represented in Fig. 342 have a *square thread*. If instead of a square a triangle is taken and is forced to displace along the cylinder in the same manner as it was done with the square, then we obtain a *screw with a*

*Screw conveyors are used for conveying corn, loose materials, etc.

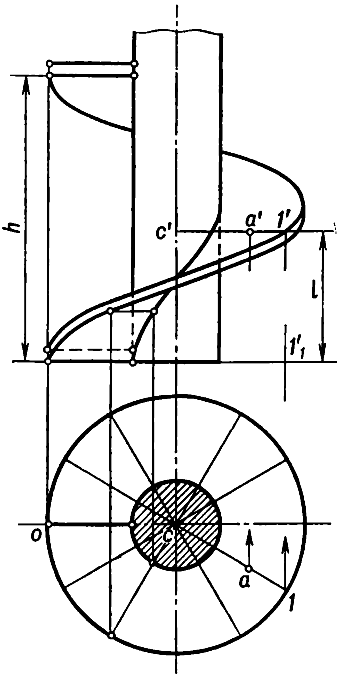

Fig. 343

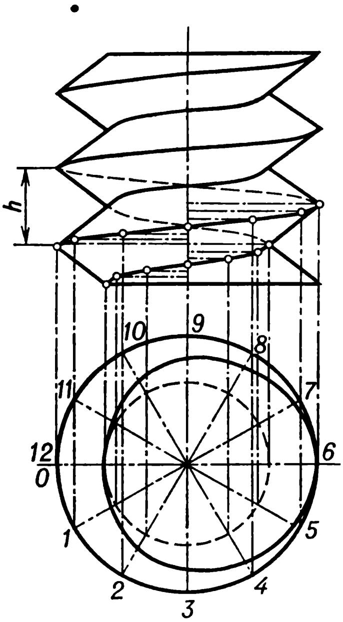

Fig. 344

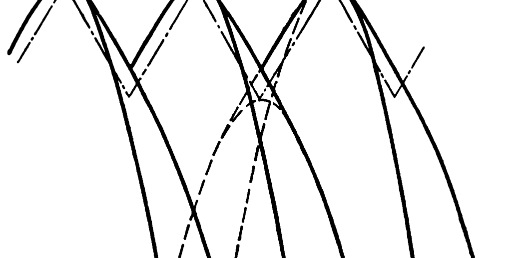

Fig. 345

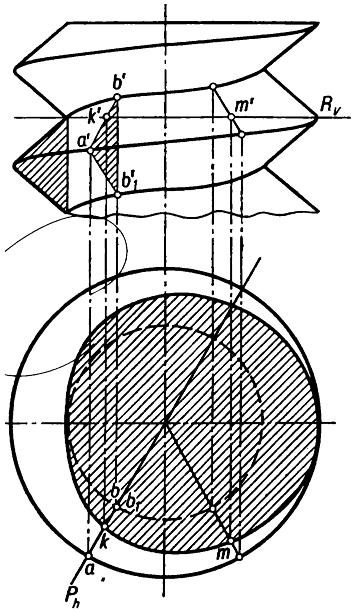

Fig. 346

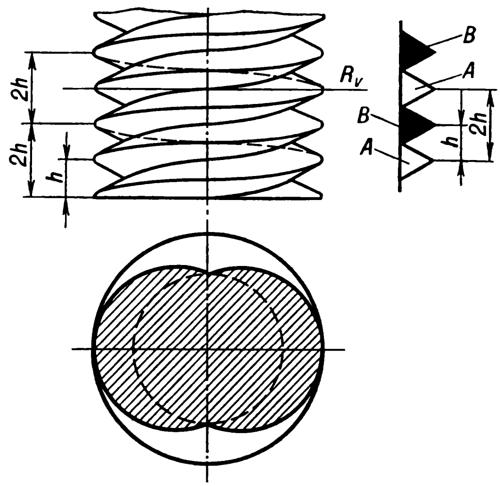

Fig. 347

triangular thread (Fig. 344). The generating triangle adjoins the cylinder with one of its sides. The vertices of the triangle generate helices which are generated with the aid of two circles. Either of the circles is divided into twelve equal parts, and the points of division are projected on the horizontal lines drawn through the twelve divisions of the screw lead. The surface of a screw with a triangular thread represents a combination of two oblique helical surfaces. The visible contour on the V plane is obtained by drawing tangents to the projections of the greater and smaller helices (Fig. 345). It is common practice, though actually the projection of an oblique helical surface on the V plane represents a curved line.

Figure 346 shows the construction of the cross-section of a screw with a triangular thread cut by a plane R. An auxiliary horizontal projecting plane P is drawn which passes through the axis of the screw. Intersecting the helical salient, the plane P singles out the generating triangle* whose horizontal projection is situated on the horizontal trace of the P plane. The vertical projection of the side AB of this triangle intersects the trace R_v at point k' which is the vertical projection of one of the points belonging to the line in which the plane R cuts the helical surface. Obtained on the

*The plane P singles out the generating triangle in two of its positions: on the front (visible) and rear (invisible) sides of the screw. Figure 346 shows the construction for the front (visible) side of the screw.

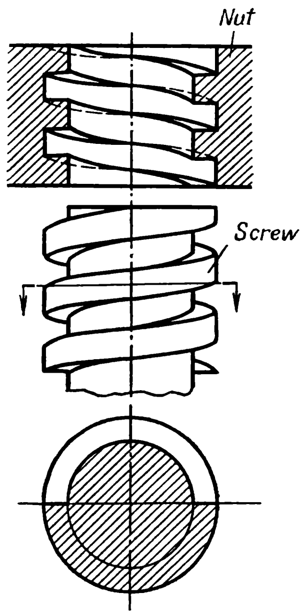

Fig. 348

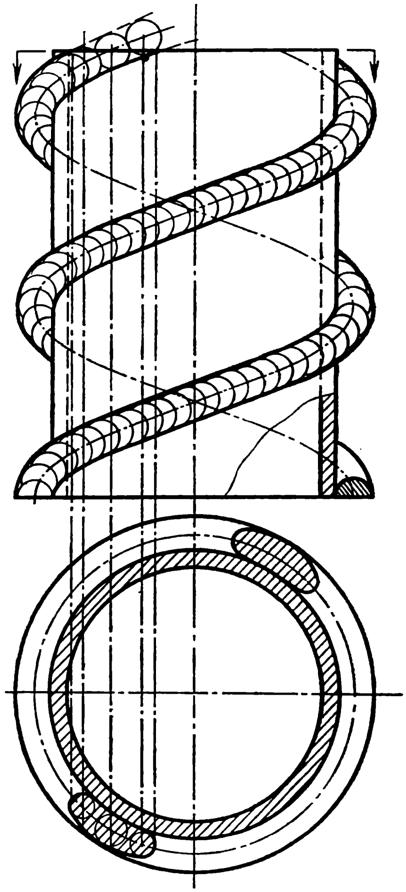

Fig. 349

line segment *ab* is the horizontal projection of the point *K* belonging to the horizontal projection of the required line of intersection of the helical surface with the plane *R*.

Furthermore, one more point $M(m', m)$ of this section is constructed. This time a horizontal projecting plane is not passed in order to show that it is quite sufficient to mark only the position of the horizontal projection of the generating triangle by drawing one of the radii. Also, instead of constructing a complete vertical projection of the generating triangle it is sufficient to construct only one of its sides, as it is shown in Fig. 346.

Drawing a number of radii and constructing the positions of the generating triangle corresponding to them, we get several points for drawing the horizontal projection of the section. As is seen, the section figure is bounded by a curved line having an axis of symmetry. Hence, we may construct only one half of the curve, the other half being drawn as a symmetrical branch. Either half of this curved line represents a spiral of Archimedes.

In the screw depicted in Fig. 344 the generating triangle, after each revolution about the axis of the main cylinder, reaches its adjacent position which is higher by the magnitude of the lead of the helix. This screw is obtained as a result of motion of one thread profile and is called a *single-thread screw.*

If a screw has two profiles which are interconnected, then we obtain a *double-thread screw.* In this case either of the profiles is lifted to the height $2h$ during one complete revolution (see Fig. 347).

A screw with a right-hand square thread complete with a nut is shown in Fig. 348. A horizontal sectional view reveals the line segments bounding the section figure together with the semi-circles. These segments testify to the fact that the thread profile is bounded not by an oblique but by a right helical surface.

A double-thread screw of a double-screw conveyor is shown in Fig. 349. This screw is formed by winding a steel circular wire on a steel tube. The wire is usually welded to the tube.

Imagining a number of spheres whose diameter is equal to the diameter of the wire and whose centres are located on the helix (the axis of the turn), we draw the contour of the projection of the turn as a line enveloping the circles which represent the projection of the spheres.

The horizontal projection shows the sections of two turns (the contour of the projection of the section is constructed as a line enveloping the circles obtained as plane sections of the above mentioned spheres).

QUESTIONS TO SEC. 52

1. How are a right and an oblique helical surfaces generated?
2. Why is a right helical surface also called the screw conoid?
3. What does an annular screw conoid represent?
4. How is a helical cylindroid generated?
5. In what lines does a plane perpendicular to the axis of the surface cut a right and an oblique helical surfaces?
6. How is it possible to approximately develop a complete turn of a right helical surface?
7. What helical surface belongs to developables?
8. What is a screw?
9. How do we make out screws with right-hand and left-hand threads by their appearance?
10. What is a multiple-thread screw?

Sec. 53. Constructing Planes Tangent to Curved Surfaces •

When representing curved surfaces and carrying out relevant constructions, we may come across a necessity to pass a plane tangent to a curved surface.

Let us take a small portion of a surface and a point on it. If through this point curves and straight lines tangent to them are drawn then the latter turn out to be contained in one plane. This plane is said to be tangent to a surface at a given point.

Points at which a unique tangent plane may be drawn are called *ordinary points*. On a surface there may be points at which it is impossible to pass a tangent plane. These points are called *singular*. They include the nodes of a surface, the points on the cuspidal edge, and the cusped vertices of surfaces of revolution (when a generatrix does not cut the axis of revolution at right angles).

A plane is quite defined by two intersecting straight lines. Therefore, to construct a plane tangent to a curved surface at one of its points, it is sufficient to draw two curves through this point on the surface and to either of them a tangent line at the same point. These two straight lines (tangents) define the tangent plane.

Finally, we introduce yet another concept, the *normal to a surface*. This is a straight line perpendicular to the tangent plane and passing through the point of tangency. It is obvious that problems on construction of normals to curved surfaces reduce, essentially, to problems on construction of tangent planes. A section by a cutting plane passing through a normal is called the *normal section*.

Figure 350 shows the construction of a plane tangent to a prolate ellipsoid of revolution at its point K. Drawn through this point are a parallel of the surface and a tangent KF to it: the projection $k'f'$ coincides with the vertical projection of the parallel, and the horizontal projection kf is tangent to the horizontal projection of the parallel which is a circle. Taken as a second curve passing through the point K is a meridian which is not shown in the drawing: we may take advantage of the principal meridian, which is already drawn representing the vertical projection of the ellipsoid. One should imagine that the ellipsoid is revolved about its axis AB so that the meridian passing through the given point K occupies the position of the principal meridian AK_1B, the point K occupying the position K_1. Drawing a tangent to the ellipse at point k'_1, we get the vertical projection of the second tangent to the ellipsoid at the point K_1. Now we have to rotate this tangent so that point k_1 occupies the initial position k. Point S being the point of intersection of the tangent and the axis of the ellipsoid, remains fixed and the tangent to the meridian at the point K will be represented by the projections sk and $s'k'$. The straight lines KF and SK define the required plane.

Obviously, the above construction is applicable to a sphere as well.

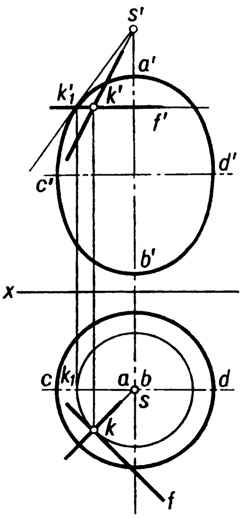

Fig. 350

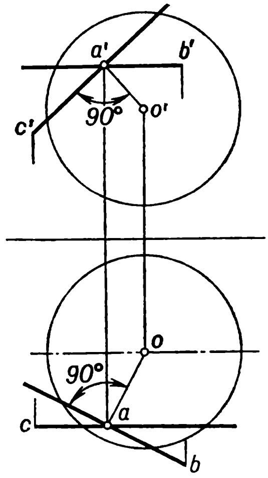

Fig. 351

But here we may proceed in a simpler way due to the fact that a *plane tangent to a sphere is perpendicular to the radius drawn to the point of tangency.* Hence, drawing the radius OA (Fig. 351), we construct the plane specifying it by a horizontal line AB and a vertical line AC which is perpendicular to OA. These lines determine the plane tangent to the sphere at its point A.

In the above considered examples (Figs. 350 and 351) the tangent plane and the surface have one point in common. If we imagine the curves on the surface passing through this point, then in the neighbourhood of the point of tangency these curves are located on one side of the tangent plane. The same could be seen on the paraboloid of revolution, on the torus generated by an arc (which is less than a semi-circle) revolving about its chord, etc. Such points on a surface are called *elliptic*. If all points of a surface are elliptic, then this surface is convex, for instance an ellipsoid shown in Fig. 350.

Figure 352 illustrates the construction of a plane tangent to a cylinder. In the left-hand drawing the plane is passed through a given point C on a cylindrical surface, in the right-hand drawing through the point K outside a cylinder.

Here the plane is tangent to a surface not at a single point but at all the points on a generating element. Such points of a surface are termed *parabolic*. Surfaces with parabolic points comprise cylindrical and conical surfaces, as also surfaces with a cuspidal edge.

The construction presented in the left-hand drawing of Fig. 352 consists in the following. We are given a ruled surface, therefore through the

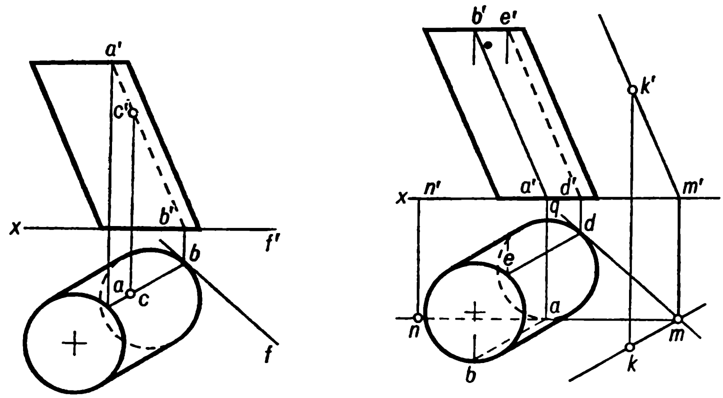

Fig. 352

point *C* we may draw a generating element *AB* which is one of the two intersecting straight lines defining the tangent plane. The tangent *BF* to a circle (which is the horizontal trace of the given cylindrical surface) may be taken as a second straight line. The lines *AB* and *BF* determine the required tangent plane, *BF* being the horizontal trace of this plane.

In the right-hand drawing of the same figure a point *K* is given outside a cylindrical surface. The tangent plane must contain a generating element of the surface. Hence, this plane is in general parallel to the direction of the generatrix. Therefore, the straight line *KM* parallel to the generatrix belongs to the tangent plane. The horizontal trace *MQ* of the required plane is taken as a second straight line. Thus, the tangent plane to the cylindrical surface is defined by two intersecting straight lines: *KM* and *MQ*. This plane touches the surface along the element *DE*.

A second solution: through the point *M* a straight line *MN* is drawn which is the horizontal trace of a second tangent plane, the element *AB* being the line of tangency.

A plane tangent to a conical surface at its point *A* is constructed in Fig. 353. The surface is specified by the vertex *S* and the directrix which is an ellipse lying on the *H* plane.

The element *SM* on which the point *A* is located is a line of tangency of a plane, tangent to the conical surface. This element together with the straight line *MN* tangent to the ellipse on the *H* plane determine the plane tangent to the given surface.

If the point through which the plane tangent to a given conical surface has to be drawn is situated outside this surface, then, to construct the tangent plane, proceed in the following way: draw a straight line through the vertex *S* and the given point, find the horizontal trace of this line, and through it draw tangents to the ellipse (as it was shown in the right-hand drawing of Fig. 352 where the tangents were drawn to a circle representing

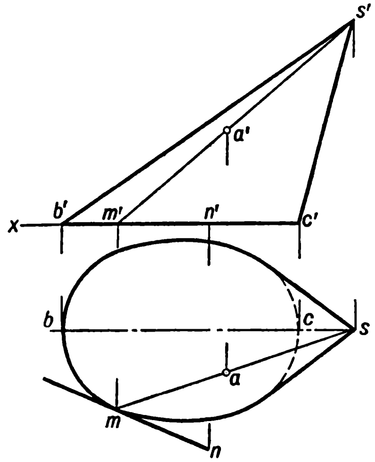

Fig. 353

the trace of the cylindrical surface on the H plane). In this case we obtain two planes tangent to the conical surface.

In the examples given in Figs. 350-353 the tangent planes do not intersect the surfaces. This is characteristic for convex surfaces. But in general a plane tangent to a surface at one of its points may intersect this surface. For instance, the plane tangent to the surface of a hyperbolic paraboloid at point O (see Fig. 321) contains the tangent lines Ox and Oy to the parabolas BOB_1 and AOA_1 and cuts the surface into two parts having with it an infinite number of points in common.

A plane tangent to a surface at one of its points may cut this surface in two straight lines intersecting at this point, in a straight line and a curve, and in two curves. For instance, the hyperboloid of revolution of one sheet, i.e. a ruled surface with two rectilinear generatrices can be cut in two intersecting straight lines. The same may be said with respect to the hyperbolic paraboloid (Fig. 321).

Nondevelopable ruled surfaces such as surfaces with a plane director and helical surfaces with a straight-line generatrix (except for an open helicoid) are cut in a straight line and a curve.

The points of a surface at which the tangent plane cuts the surface are called *hyperbolic*. Along with other surfaces (see above) we may come across such points in concave surfaces of revolution (an example of such surface is given in Fig. 330).

If a certain part of a surface consists only of hyperbolic points, then the surface has a *saddle* shape within the limits of this part (for instance the hyperbolic paraboloid, see Figs. 321 and 322).

Comparing developable ruled surfaces with nondevelopable ruled surfaces, we see that in the case of developables the tangent planes at different points of the generating line are in the same direction (for instance, in a conical surface of revolution), whereas in nondevelopables the tangent planes at different points of the generatrix have different directions (for instance, in hyperboloid of revolution of one sheet).

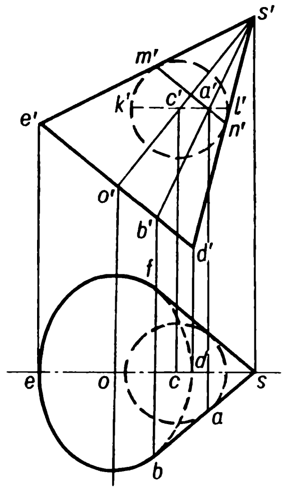

Fig. 354

Sec. 54. Examples of Constructing the Contours of the Projections of a Solid of Revolution with an Oblique Axis

Figure 354 represents a right circular cone whose axis is parallel to the *V* plane and inclined to the *H* plane. The contour of its vertical projection (an isosceles triangle *s'd'e'*) is given. It is required to construct the contour of its horizontal projection.

The required contour is formed from a part of an ellipse and two straight lines tangent to it. Indeed, the cone in its given position is projected on the *H* plane with the aid of the surface of an elliptic cylinder whose elements pass through the points on the circle of the cone base, and two planes tangent to the surface of the cone.

An ellipse in the horizontal projection can be constructed using its axes: the minor axis *de* and the major axis equal to *d'e'* (to the diameter of the circle of the cone base). The straight lines *sb* and *sf* are obtained by drawing tangent lines to the ellipse from point *s*. The construction of these lines is reduced to finding the projections of the generating elements along which the above mentioned planes touch the cone. For this purpose we use a sphere inscribed in the cone. Since a horizontal projecting plane touches both the cone and the sphere, we may draw a straight line from the point *s* tangent to the circle (which is the projection of the equator of the sphere) and take this tangent for the projection of the required element. We may begin the construction with finding the point *a'*, i.e. the vertical projection of one of the points belonging to the required element. The point *a'* is obtained as the intersection of the vertical projections of (1) the circle along which the sphere touches the cone (line segment *m'n'*) and (2) the equator of the sphere (line segment *k'l'*). Now we can find the projection *a* on the horizontal projection of the equator and draw, through the points *s* and *a*, a straight line which will be the horizontal projection of the required element. Point *B* is also determined on this line; its horizontal projection (point *b*) is the point at which the straight line touches the ellipse.

With the construction of the contours of the projections of a cone of revolution we come across, for instance, in such a case: given are the projections of the vertex of the cone (*s'*, *s*), the direction of its axis (*SK*), the dimensions of the altitude and the diameter of the base; it is required to

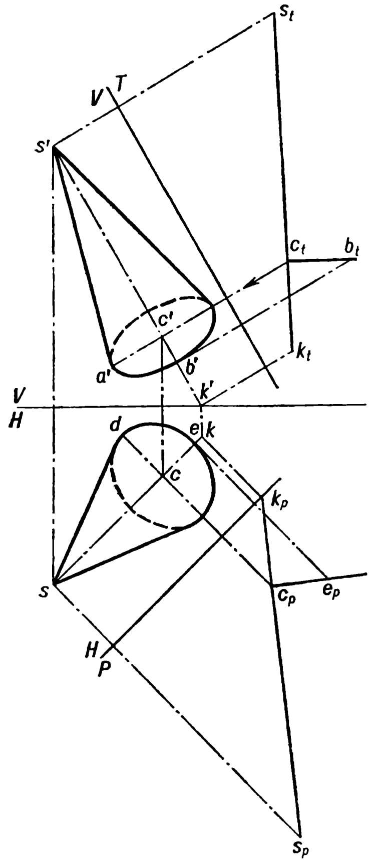

Fig. 355

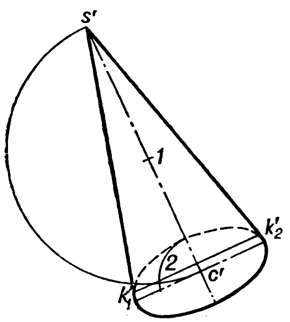

Fig. 356

construct the projections of the cone. This is done in Fig. 355 with the aid of additional planes of projection.

Thus, to construct the vertical projection, we introduce a plane *T* which is perpendicular to the *V* plane and parallel to the straight line *SK* determining the direction of the cone axis. Laid off on the projection s_tk_t is a line segment s_tc_t equal to the given altitude of the cone. At the point c_t a perpendicular is drawn to s_tc_t, and on it we lay off a line segment c_tb_t equal to the radius of the base circle. With the aid of the points c_t and b_t, we get the points c' and b' and, hence, the semiminor axis $c'b'$ of the ellipse, the latter being the vertical projection of the cone base. The line segment $c'a'$ equal to c_tb_t, represents the semimajor axis of this ellipse.

Having the axes of the ellipse, we can construct it in the way shown in Fig. 147.

To construct the horizontal projection, we introduce a projection plane *P* which is perpendicular to the *H* plane and parallel to *SK*. The construction procedure is analogous to that described for the vertical projection.

How do we construct the contours of the projections? Figure 356 shows another (compared with Fig. 354) method of drawing a tangent line to the ellipse, i.e. without using a sphere inscribed in the cone.

First, from the centre of the ellipse an arc is described of radius equal to its semiminor axis (in Fig. 356 this arc is equal to a quarter of a circle). Then the point *2* of intersection of this arc with the circle of diameter $s'c'$ is determined, and from this point a straight line is drawn parallel to the major axis of the ellipse to intersect the ellipse at points k_1' and k_2'. To complete the construction, we draw straight lines $s'k_1'$

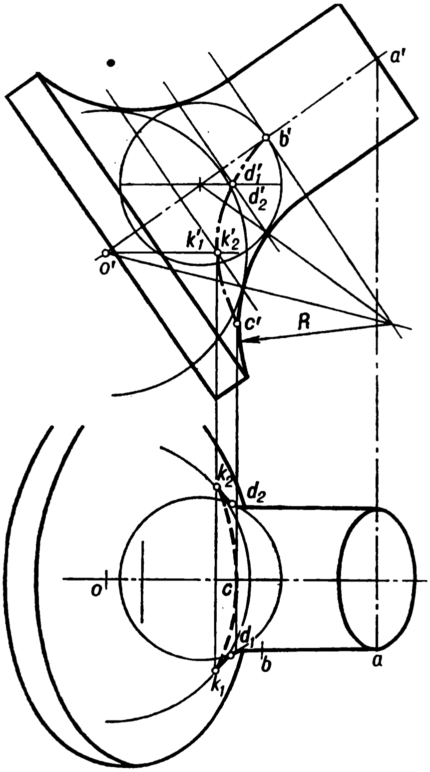

Fig. 357

and $s'k_2'$ which are tangent to the ellipse and constitute part of the contour of the projection of the cone.

Figure 357 demonstrates a solid of revolution with an inclined axis parallel to the plane *V*. The solid is bounded by a combined surface consisting of two cylinders, the surface of an annular torus and two planes. The contour of the vertical projection of this solid is its principal meridian.

The contour of the horizontal projection of the upper cylindrical part of the given solid is made up of an ellipse and two straight lines tangent to it. The straight line *ab* is the horizontal projection of the generating element of the cylinder along which the horizontal projecting plane touches the surface of the cylinder. The same refers to the contour of the projection of the lower cylinder (in Fig. 357 this contour is not completed). Now we pass over to the intermediate part of the contour which is the most complicated. We must construct the horizontal projection of the space curve through whose points there pass the projecting lines tangent to the surface of an annular torus and perpendicular to the *H* plane. The vertical projection of each point belonging to this curve is constructed using the method applied for the point *a'* in Fig. 354, i.e. with the aid of inscribed spheres. The horizontal projections of the points are determined on the projection of the equator of the corresponding sphere. In this way, for instance, point $D_1(d_1, d_1')$ is constructed.

Points k_1 and k_2 are obtained on the equator of the sphere with centre from the point $k_1'(k_2')$, the latter being obtained when constructing the line of recall tangent to the constructed curve $b'd_1'c'$.

Thus, the curve $b'd_1'k_1'c'$ contains the vertical projections of the points whose horizontal projections b, d_1, k_1 are on the contour of the horizontal projection of the solid under consideration.

QUESTIONS TO SECS. 53-54

1. What is the plane tangent to a curved surface at a given point of this surface?

2. What is a regular point of a surface?

3. How do we construct a plane tangent to a curved surface at one of its points?

4. How is the normal to a surface defined?

5. How do we construct a plane tangent to a sphere at one of its points?

6. In what case a curved surface is considered to be convex?

7. May a plane tangent to a curved surface at one of its points intersect this surface? Give an example when a surface is cut in two straight lines.

8. How do we use spheres inscribed in a surface of revolution whose axis is parallel to the V plane for constructing the outline of the projection of this surface on the H plane with respect to which the axis of the surface of revolution is inclined at an acute angle?

9. How do we draw a tangent line to an ellipse from a point lying on the extension of its minor axis?

10. In what case will the contours of the vertical and horizontal projections of a cylinder of revolution and a cone of revolution be absolutely identical?

CHAPTER 9

INTERSECTION OF CURVED SURFACES WITH A PLANE AND A STRAIGHT LINE

Sec. 55. The General Methods of Constructing Lines of Intersection of a Curved Surface with a Plane

To find the curved line of intersection of a ruled surface by a plane, we have (in the general case) to construct the points of intersection of generating elements of the surface with the cutting plane, i.e. to find the point of intersection of a straight line and a plane. The required curved line (the *section line*) passes through these points. An example is given in Fig. 358 where a conical surface specified by a point S and a curve ACE is cut by a vertical projecting plane T. The horizontal projection of the line of intersection is drawn through the horizontal projection of the points of intersection of a number of elements by the plane T.

In this example the construction is simplified due to the fact that the secant plane T is of a particular position. But the above mentioned method (obtaining the points of intersection of a number of rectilinear elements of a surface with a given secant plane in order to draw through them the required line of intersection) is applicable for any position of the cutting plane.

To construct the line of intersection of a double-curved surface by a plane, we have (in the general case) to use auxiliary planes. The points of the required line are determined as the intersections of the lines along which auxiliary planes intersect the given surface and plane. In this respect the reader should remember Fig. 166 which illustrates how auxiliary planes are used for constructing the line of intersection of two planes.

In all cases when auxiliary planes are used the constructions involved should be as simple as possible.

Figure 359 represents a solid of revolution cut by a plane specified by the trapezoid $ABCD$. To construct the points of the curved lines obtained on the surface of the solid of revolution, auxiliary secant planes are used here. Let us consider one of them, say plane Q. Intersecting the surface of the solid of revolution, this plane yields a circle (a parallel) of radius o'_q1',

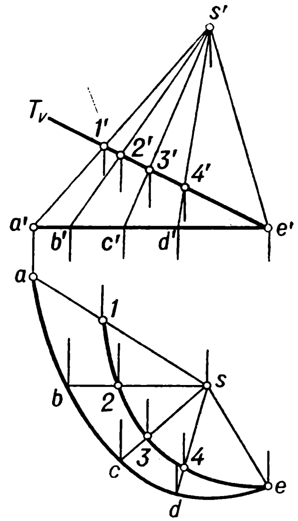

Fig. 358

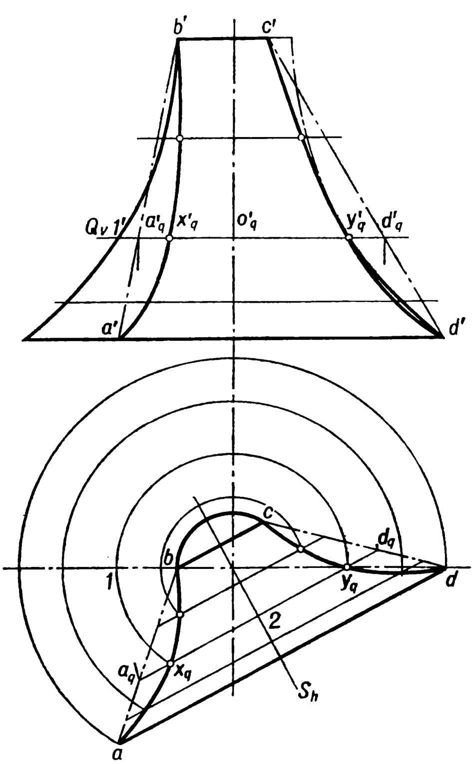

Fig. 359

and intersecting the plane *ABCD*, a horizontal line A_qD_q. The intersection of the parallel of the surface of revolution with the horizontal line A_qD_q yields the points X_q and Y_q belonging both to the surface of revolution and to the plane *ABCD*, i.e. belonging to the required line of intersection. Repeating this technique, we get a number of points determining the curvilinear part of the section line. The plane faces of the given solid of revolution are cut by the plane *ABCD* in straight lines represented by the line-segments *AD* and *BC*.

In the above considered example the construction is simplified due to the fact that the axis of the solid of revolution is perpendicular to the *H* plane, and, hence, the parallels are projected on this plane, and, hence, the parallels are projected on this plane into circles. The plane of symmetry *S* made it possible to check the correctness of the relative positions of the points belonging to the curves ax_qb and dy_qc (for instance, x_q2 should be equal to y_q2).

Applying the method of replacing projection planes or the method of revolution, we can obtain positions of the figures convenient for their construction instead of the general positions in which they were specified in the system V, H. But all this does not refer to the above described method based on the introducing of auxiliary planes. This method is applicable irrespective of the positions of the intersecting surface and plane.

In a number of cases the curve to be obtained as the intersection of a surface by a plane is known, and its projections can be constructed on the basis of their geometrical properties. Let us, for instance, recall the spiral of Archimedes (Fig. 340) obtained as the intersection of an oblique helicoid by a plane perpendicular to its axis. Obviously, it is advisable to construct this spiral as it is shown in Fig. 340 and not to find the points for it by projecting them.

Sec. 56. A Cylindrical Surface Cut by a Plane. Constructing the Development

The curved line obtained as the intersection of a cylindrical surface by a plane should be generally constructed by finding the points of intersection of generating elements with the secant plane, as it was said in the previous section with respect to the ruled surfaces in general. But this does not exclude the possibility of using auxiliary planes, each time intersecting both a surface and a plane.

Let us first of all note that *any cylindrical surface is cut by a plane parallel to the generatrix of this surface in straight lines (generating elements)*. Figure 360 shows the intersection of a cylindrical surface by a plane. In this case this surface serves as an auxiliary element for constructing the point of intersection of a curved line with a plane: through the given curve $DMNE$ (see the left-hand picture of Fig. 360) is passed a cylindrical surface which projects the curve on the H plane. Furthermore, the plane (a triangle in Fig. 360) cuts the cylindrical surface in a plane curve $M_1 \ldots N_1$. The required point of intersection of the curve with the plane (point K) is obtained as the intersection of two curves—the given and the constructed one.

This scheme of solving the problem on intersection of a curved line with a plane coincides with the scheme for solving the problem on intersection of a straight line and a plane (see Secs. 23 and 25). *In both cases through the line an auxiliary surface is drawn; for a straight line it is a plane.*

The horizontal projection of the curve $M_1 \ldots N_1$ along which the cylindrical surface intersects with the plane coincides with the horizontal projection of the curve $D \ldots E$, since this curve is the directrix for the cylindrical surface whose generating elements are perpendicular to the H plane. Therefore, using the point m_1 on the projection ac, we can find the projection m_1' on $a'c'$, and using the points n_1, the projection n_1'. Furthermore, the right-hand drawing of Fig. 360 shows an auxiliary plane S which intersects ABC along a straight line CF and the cylindrical surface along its element with the horizontal projection at point *1*. The intersection of this element with the straight line CF yields a point (with the projections *1* and *1'*) belonging

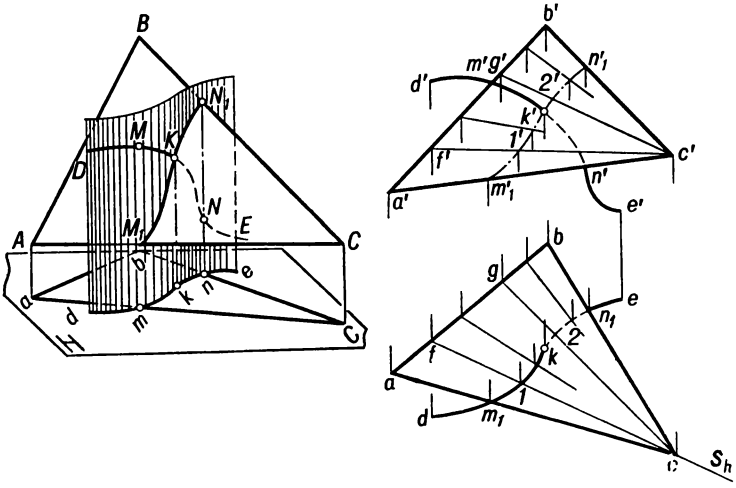

Fig. 360

to the curve $M_1 \ldots N_1$. Obviously, we may not indicate the trace of the plane but simply draw a straight line in the triangle as it is shown with respect to the line CG on which a point with the projections 2 and $2'$ is obtained.

In examples given below we are going to dwell on *developments*. A cylindrical surface may be generally developed following the scheme for developing the surface of a prism. We consider here the cylindrical surface as if it were substituted by an inscribed or circumscribed prismatic surface whose edges correspond to the elements of the cylindrical surface. The development itself is accomplished with the aid of a normal section (as in Fig. 283), but instead of a polygonal a smooth curve is drawn.

Figure 361 represents a right circular cylinder cut by a vertical projecting plane. The figure obtained in section is an ellipse with the minor axis equal to the diameter of circular base of the cylinder. The length of its major axis depends on the angle between the secant plane and the axis of the cylinder.

Since the cylinder axis is perpendicular to the H plane, the horizontal projection of the figure obtained in the section coincides with the horizontal projection of the cylinder.

To construct the points belonging to the contour of the section, we usually draw uniformly arranged elements, i.e. such elements whose horizontal projections are points equidistant from one another. This marking is convenient to be used not only for constructing the projections of the section but also for developing the lateral surface of the cylinder, as it will be shown below.

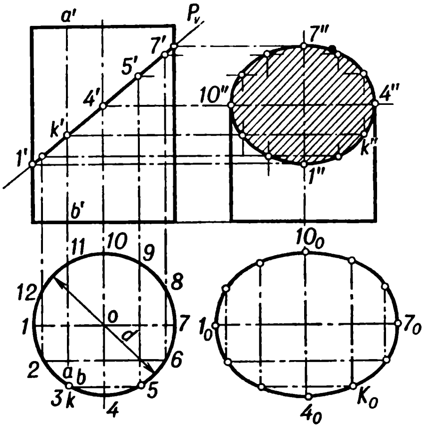

Fig. 361

The section figure is projected on the W plane into an ellipse whose major axis is equal in this case to the diameter of the cylinder and the minor axis is represented by the projection of the line segment $1'7'$. In Figure 361 the profile projection of the cylinder is constructed with the cut-off portion removed.

If in Figure 361 *the plane P was inclined to the cylinder axis at an angle of* 45°, *then the ellipse would be projected on the W plane into a circle.* In this case the line segments $1''7''$ and $4''10''$ would turn out to be equal to each other.

If the same cylinder is cut by an oblique plane inclined to the cylinder axis also at an angle of 45°, then the section figure (ellipse) can be projected into a circle on an additional projection plane which is parallel both to the axis of the cylinder and the horizontal lines of the secant plane.

Obviously, with an increase in the angle of inclination of the cutting plane to the axis the line segment $1''7''$ decreases. But if this angle is less than 45°, then the segment $1''7''$ increases to become the major axis of the ellipse on the W plane with the line segment $4''10''$ as the minor axis of this ellipse.

The true shape of the section, as it was mentioned above, is an ellipse. Its axes are obtained in the drawing: the major axis is the line segment $1_07_0 = 1'7'$, its minor axis being equal to the line segment 4_010_0 equal to the diameter of the cylinder. The ellipse can be constructed by these axes.

Figure 362 shows a complete development of the lower part of the cylinder. The developed circumference of the base circle of the cylinder is divided into equal parts in accordance with the divisions obtained in Fig. 361. The segments of the intersected elements are laid off on the perpendiculars erected at the points of division. The upper end-points of these segments correstpond to the points on the ellipse. Therefore, drawing through them a smooth

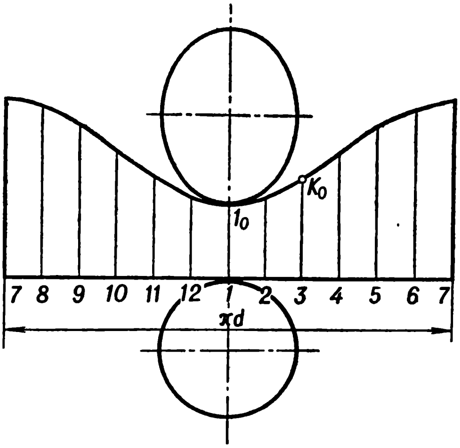

Fig. 362

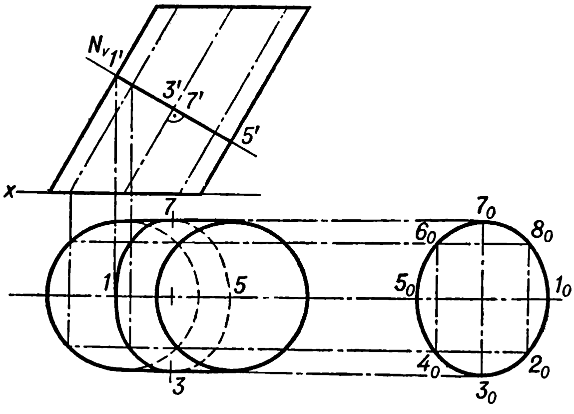

Fig. 363

curve, we get a developed ellipse (this line represents a sinusoid) which serves as the upper border of the development of the lateral surface of the cylinder.

The developed lateral surface is complete with the base (a circle) and the section (an ellipse) which enables us to make the model of a truncated cylinder.

Figure 363 demostrates an elliptical cylinder with a circular base; its axis is parallel to the V plane. To determine the normal section of this cylinder it should be cut by a plane perpendicular to generating elements, in this case by a vertical projecting plane. The normal section represents an ellipse whose major axis is equal to the line segment $3_0 7_0$, and the minor axis to $1_0 5_0 = 1'5'$.

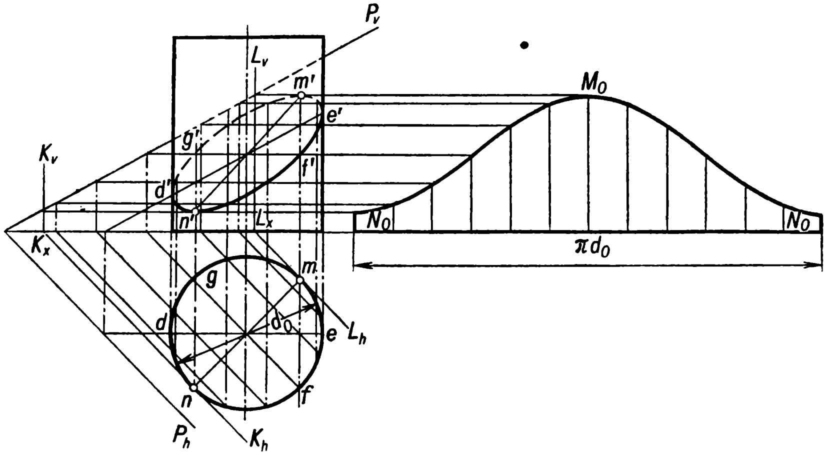

Fig. 364

If it is required to develop the lateral surface of this cylinder, then we develop the curve bounding the normal section into a straight line, erect perpendiculars to it at appropriate points of this line, and lay off on them the line segments of the generating elements taking them from the vertical projection. To mark the elements we divide the circumference of the base circle into equal parts. As a result, the ellipse (the normal section) is also divided into the same number of parts but not all of them are of equal length. The ellipse can be developed into a straight line by consecutively laying off sufficiently small parts of the ellipse.

Figure 364 shows a right circular cylinder cut by an oblique plane. The secant plane forms an acute angle with the cylinder axis and, therefore, cuts the cylinder in an ellipse.

Like the case represented in Fig. 361, the horizontal projection of the section coincides with the horizontal projection of the cylinder. That is why the position of the horizontal projection of the point of intersection of any element of the cylinder with the plane *P* is known (for instance, point *a* in Fig. 365). To find the corresponding vertical projection we may draw in the plane *P* a horizontal or a vertical line on which the required point must be located. Figure 365 uses a vertical line. The projection *a'* lies at the point where the vertical projection of the vertical line intersects the vertical projection of the corresponding element. One and the same vertical line determines two points of the curve, *A* and *B* (Fig. 365). But if a vertical line corresponding to point *C* is constructed, then this line will determine only one point of the curved line of intersection. The vertical line drawn through the points *D* and *E* determines the extreme points *d'* and *e'*.

Continuing analogous constructions, we may find sufficient points for drawing the vertical projection of the line of intersection.

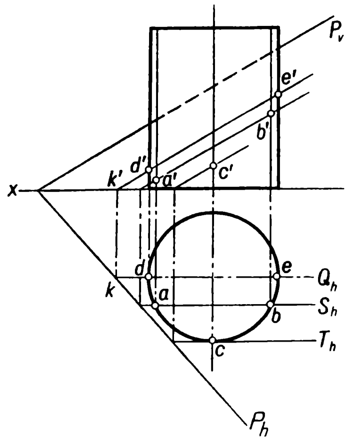

Fig. 365

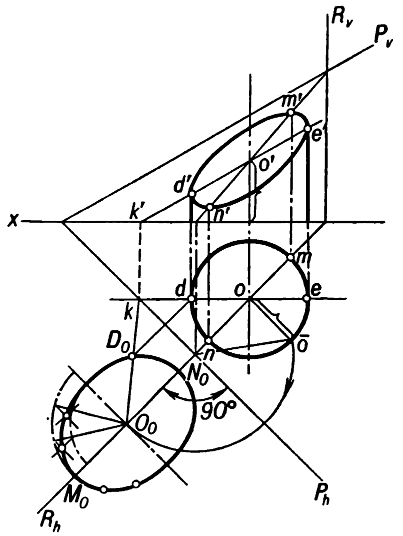

Fig. 366

The upper part of the cylinder shown in Fig. 366 is cut away. If it is not removed, then the vertical projection represents a complete cylinder, and the line of intersection is drawn as it is shown in Fig. 364.

Figure 365 shows auxiliary vertical planes Q, S, T intersecting the cylinder along generating elements, and the plane P, along vertical lines. This corresponds to what was said at the beginning of the present section. The auxiliary plane T only touches the cylinder, thus determining only one point on the curve.

When constructing the vertical projection of the line of intersection, besides the points d' and e' (Fig. 365), we have to find two more reference points, namely, m' and n', i.e. the uppermost and lowest points of the vertical projection of the section. For their construction, we have to take an auxiliary plane perpendicular to the trace P_h and passing through the axis of the cylinder (Fig. 366). This plane is a common plane of symmetry for the given cylinder and secant plane P. On finding the line of intersection of the planes P and R, we mark the points m' and n' by constructing them on the vertical projection with the aid of the points m and n.

Another method for finding the points m' and n' consists in passing two planes tangent to the cylinder whose horizontal traces are parallel to the trace P_h. These planes will intersect the plane P along its horizontal lines (auxiliary planes K and L in Fig. 364). On marking the points m and n, we construct the points m' and n' on the vertical projection of the found horizontal lines.

The line segment MN represents the major axis of the ellipse which is the figure cut by the plane P from the given cylinder. This is seen in Fig. 366,

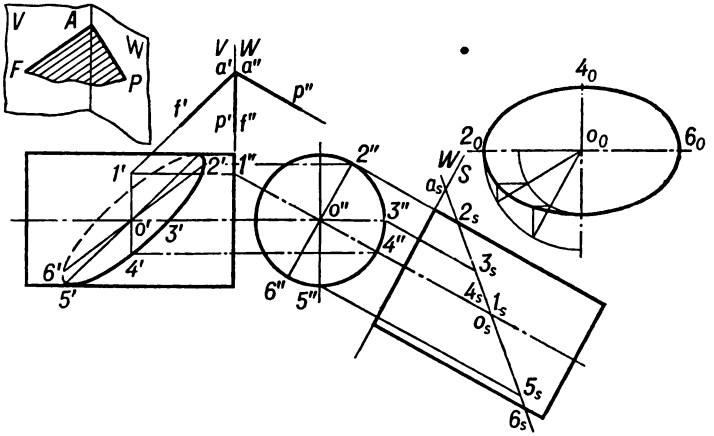

Fig. 367

where the true size of the section is constructed (an ellipse coincident with the H plane). But the line segment $m'n'$ in the same drawing is by no means the major axis of the ellipse which is the vertical projection of the section figure. The major axis can be found with the aid of the conjugate diameters $m'n'$ and $f'g'$ (Fig. 364) using the construction indicated in Sec. 21, or a special construction set forth in Sec. 76.

The true size of the section can be found by bringing the secant plane into coincidence with one of the projection planes, H or V.

In Figure 366 the ellipse in the coincident position is constructed using its major and minor axes (in the same drawing point D_0 is obtained by bringing into coincidence a vertical line).

The development of the lateral surface is shown in Fig. 364. Pay attention to the fact that the points (horizontal projections of the elements) are marked on the circumference of the base beginning with the point n. This simplified the construction, since one and the same horizontal line yielded two points on the vertical projection of the ellipse. Besides, the development has an axis of symmetry. But the points d and e are not among the points marked on the circumference.

Another example of constructing the section of a cylinder of revolution by a plane is given in Fig. 367. This construction is carried out using the method of replacing projection planes. The secant plane is specified by two intersecting straight lines: a vertical line (AF) and a profile line (AP). Since the profile projection of the vertical line and the vertical projection of the profile line lie on one line ($a' \equiv a''$, $a''f'' = a'p'$), these straight lines are contained in the planes V and W, respectively (see the left-hand picture of Fig. 367). The axis V/W passes through $a''f''$ ($a'p'$).

We introduce a new plane S so that S is perpendicular to W and AP. The secant plane turns out to be perpendicular to S, and the projection of the section figure on S is obtained in the form of a line segment 2_s6_s equal to the major axis of the ellipse. The position of the segment a_s6_s is determined by constructing the projections of the points A and 1 on the plane S.

Let us follow the construction of some points. To avoid unnecessary constructions, the projection $1''$ was taken on the extension of the perpendicular drawn from o'' onto W/S. The projection $1'$ was obtained from the point $1''$; the line segment $1'1''$ laid off from the axis W/S determined the point 1_s and the projection of the centre of the ellipse (point o_s) which coincides with the point 1_s. Knowing the projections o_s and o'', we can obtain the centre of the ellipse o' which is the required vertical projection of the section figure.

Points 2_s and $2''$ determined the point $2'$ which is the nearest to the plane W, and the points 6_s and $6''$ the point $6'$—the most remote from W.

Point 5_s is taken from $5''$, and found with the aid of points 5_s and $5''$ is the point $5'$ which is one of the points determining the separation of the ellipse on the vertical projection of the cylinder into visible and invisible parts. The second point is located symmetrically to the point $5'$ about o'.

The rest of the constructions is obvious from the drawing. The section figure in its true size (the ellipse in the top right corner of Fig. 367) is constructed by its axes: the major axis is equal to 2_s6_s, the minor axis being equal to the diameter of the cylinder.

QUESTIONS TO SECS. 55-56

1. How do we construct the curved line of intersection of a curved surface by a plane?

2. In what lines is a cylindrical surface cut by a plane passed parallel to the generatrix of this surface?

3. What method is generally used for finding the point of intersection of a curved line and a plane?

4. In what lines is a cylinder of revolution cut by planes?

5. In what case is the ellipse obtained as the intersection of a cylinder of revolution whose axis is perpendicular to the H plane by a vertical projecting plane projected on the W plane into a circle?

6. How an additional plane of projection should be arranged so that the ellipse obtained as the intersection of a cylinder of revolution whose axis is perpendicular to the H plane and an oblique plane inclined to the cylinder axis at an angle of 45° is projected on this additional plane into a circle?

Sec. 57. A Conical Surface Cut by a Plane. Constructing the Development

To construct the curved line obtained as the intersection of a conical surface with a plane, in the general case the points of intersection of generating elements with the secant plane should be found.

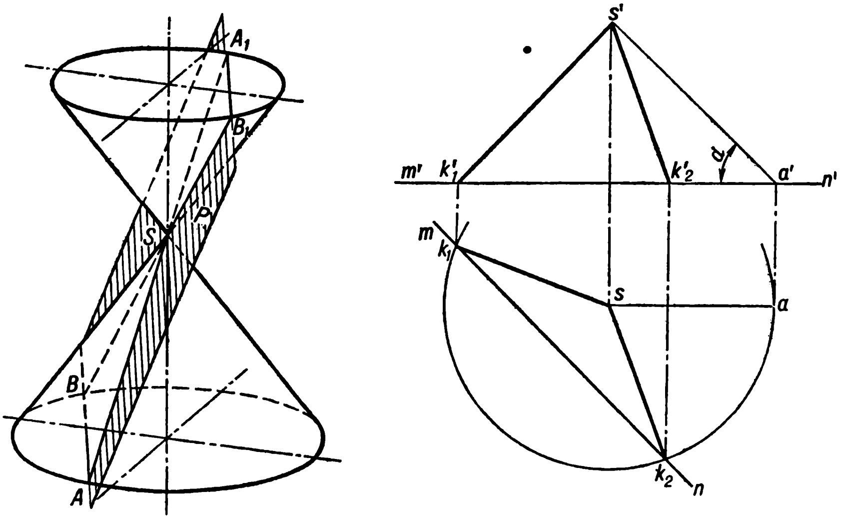

Fig. 368

If a plane cutting a conical surface passes through its vertex, then two straight lines (generating elements) are obtained (AA_1 and BB_1 in Fig. 368).

Let us consider an example illustrating such intersection of a conical surface.

Suppose that in a plane specified by a point S and a horizontal line MN (Fig. 368, right) it is required to draw through the point S a straight line forming an angle α with the H plane.

The locus of straight lines forming an angle α with the H plane is a conical surface of revolution whose axis is perpendicular to the H plane and with the point S as its vertex (by hypothesis). Consequently, the given plane passes through the vertex of a cone and cuts its surface in straight lines (elements). These elements are the required lines, since they pass through the point S in a given plane at a given angle α to the H plane.

Now we have to represent the cone (it is not completed in Fig. 368), for which purpose we draw a straight line $s'a'$ and describe from s as centre an arc of radius sa, the base of the cone being taken in the horizontal plane passing through the line MN. The rest is obvious from the accompanying drawing.

Compare this construction with those accomplished in Figs. 245 and 246 (see Sec. 38).

Figure 369 represents a right circular cone placed on the H plane. Q is a tangent plane to the given cone, it touches the cone along the generating element SC. The trace Q_h is tangent to the circle which is the horizontal projection of the cone base. The fact that the point S lies in the plane Q is established with the aid of a horizontal line SN. The plane P passes through the

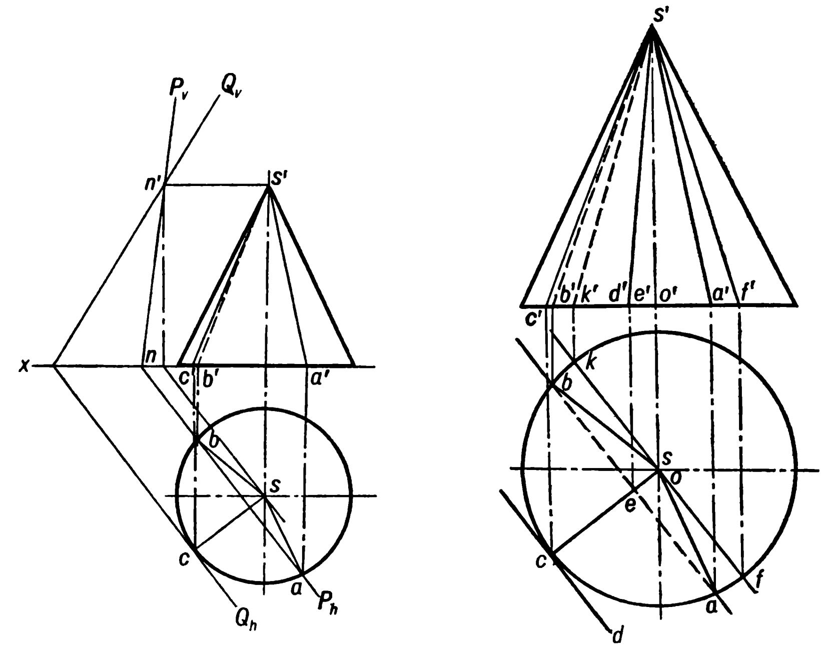

Fig. 369

vertex of the given cone and intersects the latter along the elements *SA* and *SB*.

The right-hand drawing of the same figure represents the planes not by their traces. The plane tangent to the cone is specified by the element *SC* and a straight line *CD* tangent to the base circle. The plane passing through the vertex and intersecting the cone along the elements *SA* and *SB* is specified by a straight line *AB* contained in the plane of the cone base and a straight line *SE* passing through the vertex of the cone and intersecting the line at point *E*.

If a plane passes through the cone axis, then it cuts the cone in two elements with an angle between them maximal for the given cone. In the mentioned drawing these elements are *SF* and *SK*, the angle between them being equal to the vertex angle between the contour lines of the vertical projection of the cone.

If the secant plane does not pass through the vertex of the cone then the latter is cut in one of the following four curves: (1) an *ellipse* if the secant plane intersects all the elements of one nappe of the conical surface, or in other words, it is parallel to none of the cone elements (planes Q, Q_1, and Q_2 in Fig. 370); in this case the angle between the secant plane and the cone axis is greater than that between this axis and the generatrix of the cone; (2)

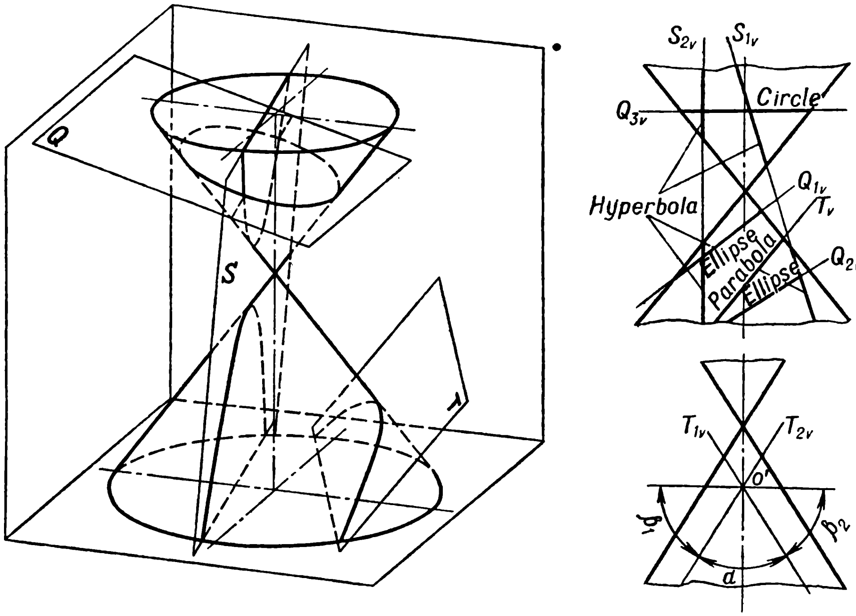

Fig. 370

a *circle** if the secant plane is perpendicular to the axis of the cone (the plane Q_3 in Fig. 370); (3) a *parabola* if the secant plane is parallel only to one of the elements (the plane T in the same figure); in this case the angle between the secant plane and the cone axis is equal to the angle between this axis and the generatrix of the cone; (4) a *hyperbola* if the secant plane is parallel to two elements (the planes S, S_1, and S_2 in the same drawing); in this case the angle between the secant plane and the cone axis is less than that between this axis and the generatrix of the cone.

The right-hand (bottom) drawing presents the angles α, β_1, and β_2, α being the angle between the traces T_{1v} and T_{2v} of the planes cutting the cone in parabolas. If the traces are drawn through the point o' inside the angle α, then this determines the planes cutting the cone in hyperbolas, and if they are drawn through the point o' inside the angles β_1 and β_2, then we obtain the planes cutting the cone in ellipses.

Consider the proof of the assertion that *if the secant plane is parallel to none of the cone elements and does not pass through its vertex, then the cone is cut in an ellipse.*

Whatever the given relative positions of the cone and secant plane in space, by transforming the drawing they can always be brought to a position in which the axis of the cone will turn out to be perpendicular to the H plane,

*May be regarded as an ellipse with equal axes.

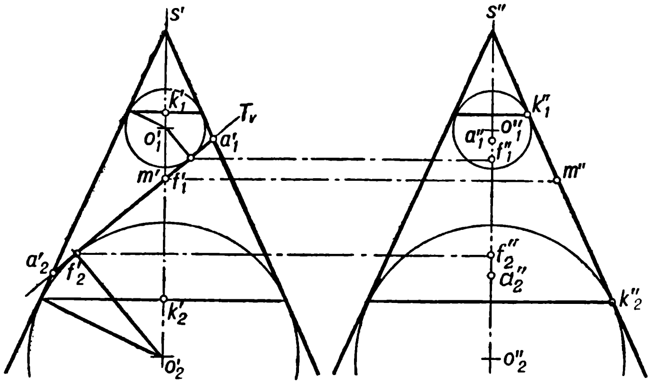

Fig. 371

and the secant plane will become a vertical projecting plane. Figure 371 shows a cone and a secant plane T in just such relative positions represented in the vertical and profile projections.

Inscribed in the cone are spheres tangent to the plane T at points F_1 and F_2, and to the cone along parallels passing through points K_1 and K_2, respectively. The points F_1 and F_2 are obtained in the plane of the principal meridian and, consequently, are collinear with the points A_1 and A_2 belonging to the section of the cone by the plane T. This section is projected on the V plane into a line segment $a_1'a_2'$.

Let us consider the generating element of the cone lying in the profile plane and mark on it the points K_1 and K_2 at which the inscribed spheres touch this element, and the point M belonging both to the same element and to the curved line of intersection of the cone by the plane T. It is known that the segments of tangents drawn from a point to a sphere determined by this point and the points of tangency are equal to each other. Whence, $MK_1 = MF_1$ and $MK_2 = MF_2$. Adding termwise, we get $MK_1+MK_2 = = MF_1+MF_2$. But $MK_1+MK_2 = K_1K_2$, i.e. the sum of distances of a point taken on the section curve to two fixed points F_1 and F_2 belonging to the plane containing this section is a constant equal in this case to the line segment K_1K_2. This segment of a cone element is situated between two of its parallels and does not depend on the choice of the point M on the section curve. Indeed, if not M but another point were taken on the section curve of the cone, then the element passing through it would touch both spheres at points lying on the same parallels. The segment of that element between the points of tangency would be equal to the same segment K_1K_2.

The conclusion drawn shows that the point M belongs to the locus of points the sum of whose distances from two given points has a certain constant value. This corresponds to the definition of the *ellipse*.

Similarly, we draw conclusions for the cases of intersection of the cone of revolution along a parabola and hyperbola.

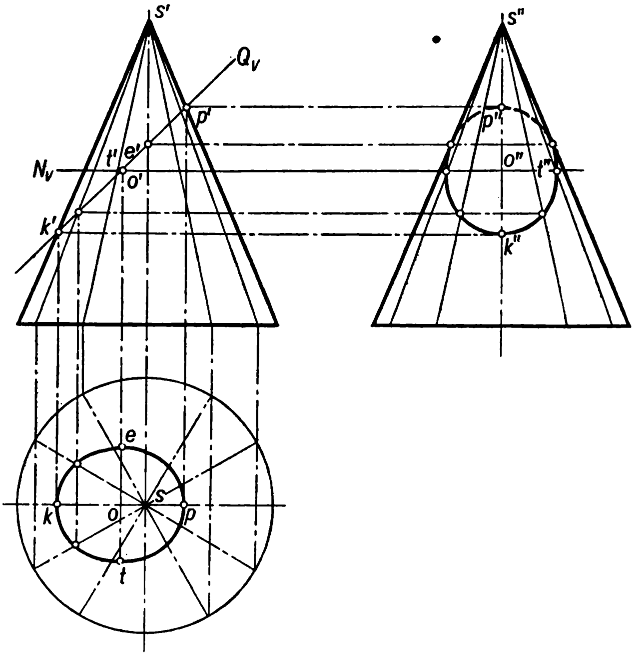

Fig. 372

Figure 372 represents a cone of revolution cut by a vertical projecting plane. The points of intersection of the trace Q_v with the vertical projections of elements represent the projections of points belonging to the required curve of intersection (in the present case it is an *ellipse*). With the aid of these projections, the horizontal and profile projections are found.

One of the axes of the ellipse (the major one) is projected on the V plane into a line segment $k'p'$; the other (minor) axis perpendicular to the V plane is projected into a single point, i.e. into the mid-point of the segment $k'p'$.

If through the point O a plane N is passed perpendicular to the cone axis (in the case parallel to the H plane), then the projection of the minor axis is obtained (Fig. 373) as the chord te of a circle which is the horizontal projection of the conic section by the plane N.

The projection of the minor axis can also be obtained by the construction shown in the right-hand drawing of Fig. 373. The cone is cut in a triangle revolved and brought in coincidence with the V plane. The line segment o_0t_0 is equal to the semiminor axis. Laying off this segment from the point o perpendicular to kp, we obtain the minor axis ($tt_1 = 2o_0t_0$).

The horizontal and profile projections of the section figure are ellipses. The profile projection may turn out to be a circle, since at a certain inclination of the secant plane the projections of the axes of the ellipse may appear

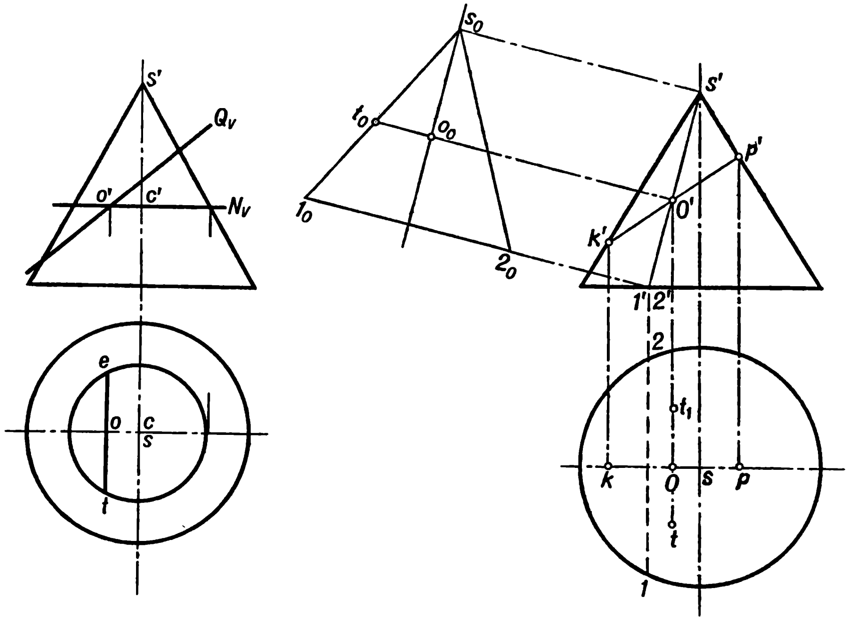

Fig. 373

equal to each other. But the projection of the section figure (ellipse) on a plane perpendicular to the cone axis (in the present case on the *H* plane) cannot be a circle.

Figure 374 (left) shows how to find for a cone the direction of the vertical trace of vertical projecting planes cutting this cone in ellipses which are projected on the *W* plane into circles. The construction is carried out on the vertical projection of the cone. The bisector of the angle $s'm'k'$ intersects the axis of symmetry at point n'. Drawing at this point a perpendicular to the bisector $m'n'$, we find the point p'. A straight line drawn through the points k' and p' indicates the direction for the vertical traces of the required secant planes. The problem is reduced to constructing the diagonal of an equilateral trapezoid $k'm'p'q'$ in which a circle with centre at the point n' can be inscribed. Drawing through the point n' a straight line parallel to $q'p'$, we get the point o', i.e. the projection of the centre of the ellipse whose vertical projection is the line segment $k'p'$.

Will the ellipse obtained as a result of intersection of the cone with the plane *Q* be projected on the *W* plane in the form of a circle? (See the right-hand drawing of Fig. 374). The construction carried out in Fig. 374 yields one of checking methods: through the point p' we draw a straight line parallel to the base, construct the bisector of the angle $p'q'k'$, and get the point n'. Since the perpendicular drawn at point n' to this bisector does not

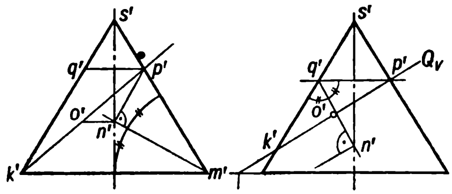

Fig. 374

pass through the point k', the profile projection of the section figure will turn out to be an ellipse but not a circle.

Figure 375 shows the construction of the vertical projection of a hyperbola obtained as a result of intersection of a cone of revolution with a horizontal projecting plane.

Since the horizontal projection of the hyperbola coincides with the trace S_h, the intersection of this trace with the horizontal projection of the base determines the points a and b. Using these points, we find the projections a' and b'.

To find the point c' (the uppermost point of the vertical projection of the hyperbola), we draw an auxiliary horizontal projecting plane through the cone axis perpendicular to the trace S_h. The horizontal projection c of the required point C is obtained as the intersection of S_h and T_h. On finding the vertical projection of the element SK, we mark the point c' on it.

We then determine the point d' at which the vertical projection of the hyperbola is separated into a visible and invisible parts. This point is found with the aid of the generating element SN.

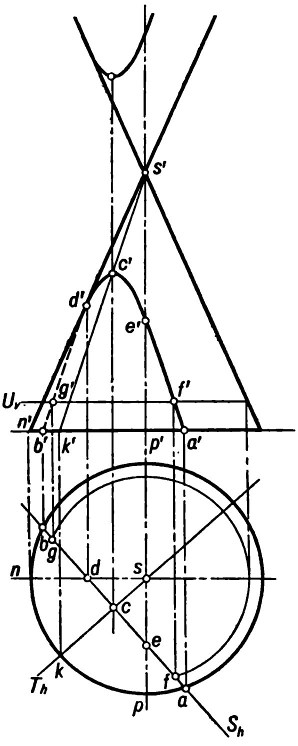

Fig. 375

Other points of the hyperbola can be found by drawing several elements within the limits of the portion of the conical surface denoted by the letters $SAKB$, or several auxiliary secant planes. Shown in Fig. 375 is one of such auxiliary planes—a horizontal plane U cutting the surface of the cone in a circle. With the aid of this plane points F and G are found.

A second hyperbola is obtained on the second nappe of the conical surface.

Figure 376 shows the construction of the projections of a section of a right circular cone by an oblique plane specified by a horizontal line AC and a vertical line AB. Furthermore, the section figure is constructed in itstrue size.

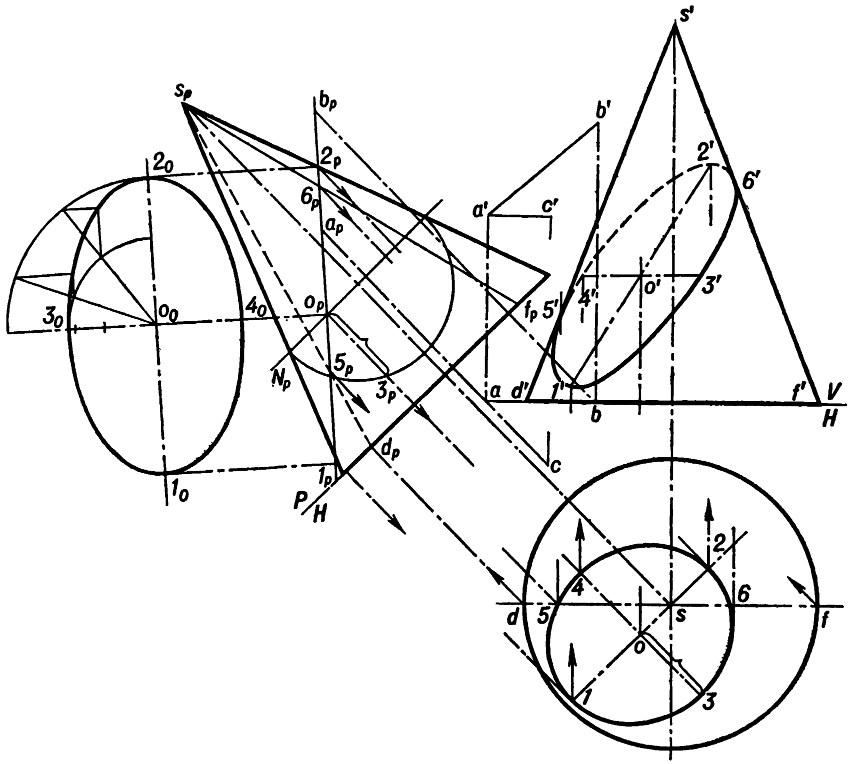

Fig. 376

The construction is carried out using the method of replacing projection planes. An additional projection plane P is introduced chosen so that it is perpendicular not only to the H plane but also to the secant plane: the axis P/H is drawn perpendicular to the projection ac. The secant plane is projected on the plane P into a straight line on which the section figure (the segment 1_p2_p) is located. This determines the major axis of the ellipse in which the cone is cut by the given plane. The projection of the centre of the ellipse is found at the point o_p bisecting the segment 1_p2_p. The plane N passed perpendicular to the axis of the cone makes it possible to find the minor axis of the ellipse (in Fig. 376 a semi-circle is described and inside it a line segment o_p3_p is constructed equal to half the minor axis of the ellipse). Using the points o_p, 1_p, 2_p, we find the horizontal projections o, 1, 2, and then the vertical projections o', $1'$, $2'$ situated at the same distance from the axis V/H at which the horizontal projections o_p, 1_p, 2_p are located from the axis P/H. $2'$ is the uppermost point on the vertical projection, and $1'$ is the lowest point of the ellipse which represents the vertical projection of the section figure. To determine the positions of the points $5'$ and $6'$ at

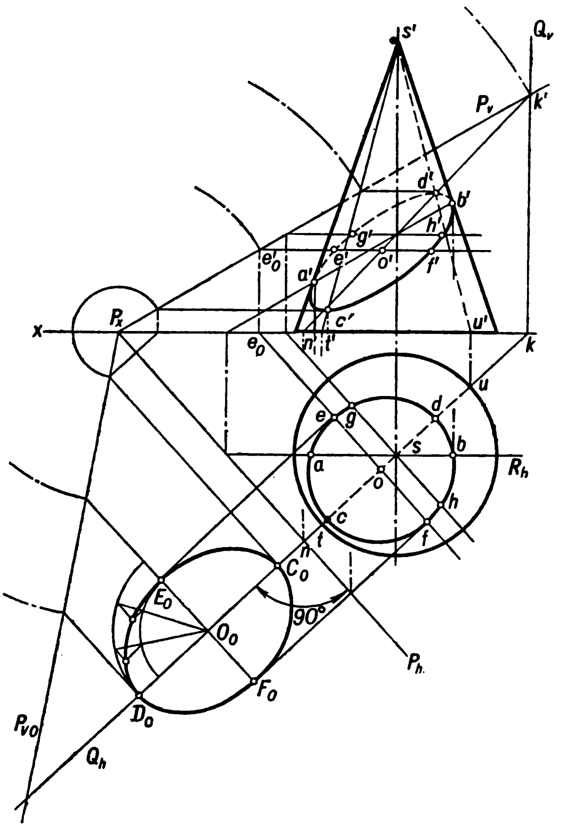

Fig. 377

which the ellipse on the vertical projection is separated into "visible" and "invisible" parts, we construct the projections s_pd_p and s_pf_p of the generating elements SD and SF, and find the points 5_p and 6_p. Using the latter, we construct the horizontal projections 5 and 6, and then the vertical projections $5'$ and $6'$. But it is possible to find only the point $5'$ and to draw through it a straight line parallel to the projection $a'b'$, since the plane containing the principal meridian intersects the given secant plane along a vertical line.

Being situated on the horizontal of the secant plane, the minor axis of the ellipse is projected on the H plane true length (segment 3-4) and serves also as the minor axis for the ellipse which is the horizontal projection of the section figure. The true size of this figure is obtained by constructing an ellipse by its major axis ($1_02_0 = 1_p2_p$) and minor axis ($3_04_0 = 3$-4)

Figure 377 demonstrates an analogous construction with the secant plane specified by its traces.

We begin the construction with finding the points lying on the contour of the vertical projection of the cone. To this end we pass through the cone axis an auxiliary secant plane R parallel to the V plane, R_h being the trace of this plane. The plane R intersects the plane P along a vertical line and the cone along two elements. The points A and B yielded by the intersection of the vertical line with the elements of the cone belong to the required line of intersection of the cone by the plane P.

At points a' and b' the vertical projection of the line of intersection touches the contour of the vertical projection of the cone and is separated into two parts: visible and invisible. Then another two reference points* are constructed, namely, the uppermost and lowest points of the section for which purpose we draw an auxiliary secant plane Q (a horizontal projecting one) perpendicular to the trace P_h and passing through the axis of the cone. The plane Q cuts the cone along the elements ST ($s't'$, st) and SU ($s'u'$, su) and the plane P along the line NK ($n'k'$, nk). Points C and D obtained as the intersections of the elements ST and SU with the line NK are the required points. The line segment CD is the major axis of the ellipse in which the given cone is cut by the plane P. The projection cd is the major axis of the ellipse which is the horizontal projection of the section figure. Bisecting CD, we get the position of the centre of the ellipse; points o' and o are the centres of the ellipses (which are the projections of the section figure).

To find intermediate points of the line of intersections, it is convenient to use horizontal secant planes, since they cut the surface of the cone in circles and the plane P along horizontal lines. Suitable for this construction are only the planes whose vertical traces are contained within the limits of c' and d', since in this case there are no points higher than the point d' and lower than the point c' belonging to the line of intersection. Figure 377 shows the construction of points E, F, G, H with the aid of two such planes. One of them is passed through the point O thus determining the line segment ef which represents the minor axis of the ellipse obtained as a result of intersection of the cone by the plane P, and at the same time the minor axis of the horizontal projection of this ellipse.

The line segments $c'd'$ and $e'f'$ are conjugate diameters** for the ellipse representing the vertical projection of the section figure. Using them, we can find the axes of the ellipse.

The true size of the section is found by bringing the secant plane into coincidence with the H plane. An ellipse can be constructed by its axes whose lengths are found by bringing into coincidence the end-points of the axes: C_0 and D_0 for the major axis, and E_0 and F_0 for the minor axis.

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*Such points as the farthest from and the nearest to the plane of projection; those separating a curve into a visible and invisible parts, the end-points of the axes of ellipses are called reference points.

**For conjugate diameters see Sec. 21.

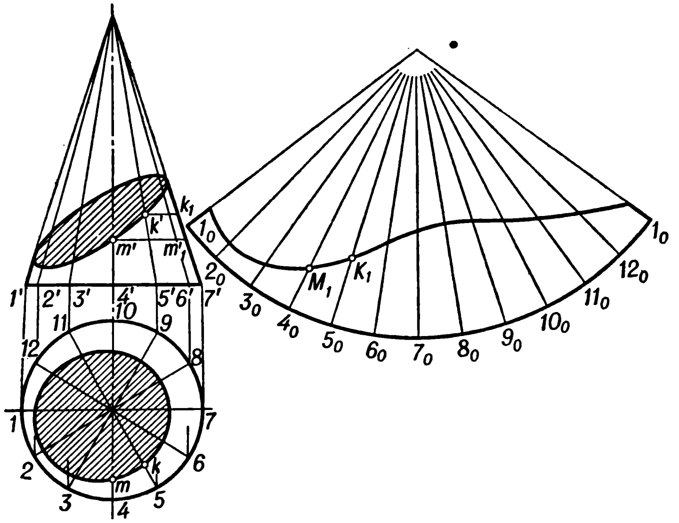

Fig. 378

Figure 378 shows the construction of the development. The lateral surface of the cone is developed into a circular sector whose angle is computed by the formula $\alpha = \frac{r}{l} \cdot 360°$, where r is the radius of the circumference of the cone base, and l is the generatrix of the cone.

To construct the line of intersection on the developed lateral surface of the cone, we have to draw a number of generating elements and to determine the lengths of their segments. Then we construct these elements on the developed surface of the cone and lay off the lengths of the corresponding segments. Figure 378 illustrates the development of the lateral surface of the cone with the line of intersection constructed on it. The lengths of segments of the relevant elements are determined by revolving the elements to the position parallel to the V plane (this construction is shown for two elements).

Figure 379 indicates how the points farthest from and nearest to the plane H are found to construct the line of intersection. In this case a cone is cut by an oblique plane Q. To construct these points planes P and T are passed tangent to the cone so that their traces P_h and T_h are parallel to Q_h. This determines the elements of the conical surface on which the required points K and M must be located.

First we construct the horizontal projections k and m as the points of intersection of the horizontal projections of the horizontal lines along which the planes P and T intersect the plane Q with the horizontal projections of the elements SA and SB, and then the projections k' and m' are marked on the vertical projections of these elements.

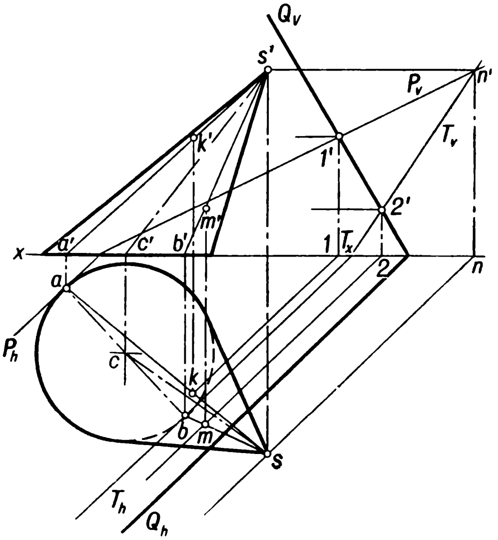

Fig. 379

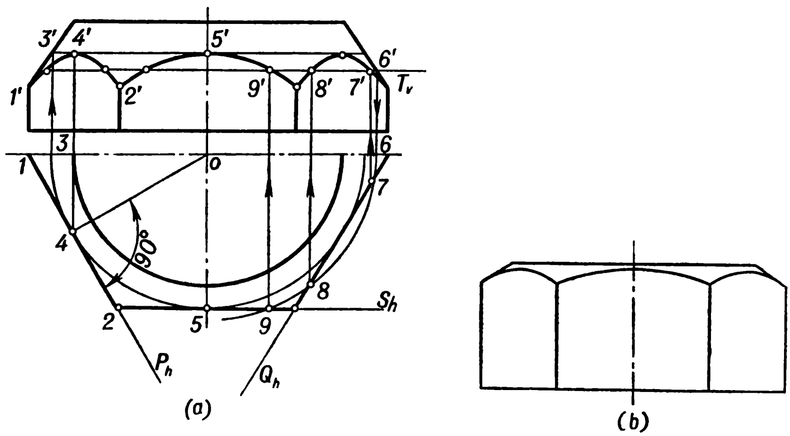

Fig. 380

Figure 380, *a* shows the construction of the curves obtained on the surface of a cone of revolution when intersected by the faces of a regular hexagonal prism*. Two of the lateral faces are situated in horizontal projecting planes *P* and *Q*, and the third one in a vertical plane *S*. The positions of these planes relative to the axis of the cone enable us to determine at once

*To save the space, only half of the horizontal projection is represented.

what curves are obtained as the intersections. In this case we obtain hyperbolas, one of them being projected on the V plane without distortion.

To find the points of the curves, parallels are taken on the cone. First of all reference points *1*, *4*, *2*, *5* are found on the horizontal projection. From them we determine the points *1'*, *4'*, *2'*, *5'* on the vertical projection. Then, with the aid of an auxiliary horizontal plane T, first point *6'* is determined on the contour of the vertical projection of the cone, and then point *6* is obtained. Using a circle of radius *o6* we construct points *7*, *8*, and *9*, and finally find the points *7'*, *8'*, *9'*.

Figure 380, *b* represents a hexahedral nut (only the front view is given). The curves separating the lateral faces of the nut from its conical part represent hyperbolas; their projections are constructed in the way similar to that shown in Fig. 380, *a*.

QUESTIONS TO SEC. 57

1. What is the general method of constructing the curved line of intersection of a conical surface by a plane?
2. How a plane should be passed in order to cut a conical surface in straight lines?
3. In what curves is a cone of revolution cut by planes?
4. Is it possible to inscribe spheres in any conical surface?
5. How do we construct the minor axis of an ellipse obtained as the intersection of a cone of revolution by a plane?
6. Into what curve is an ellipse obtained as the intersection of a cone of revolution projected on a plane perpendicular to the axis of the cone?
7. How do we construct the development of the lateral surface of a cone of revolution?
8. What curves are represented on a nut with a conical chamfer?

Sec. 58. A Sphere and a Torus Cut by a Plane. An Example of Constructing the Line of Intersection on the Surface of a Combined Solid of Revolution

Whatever the direction of the secant plane is, the latter always cuts a sphere in a circle which is projected into a line segment, an ellipse, or a circle depending on the position of the secant plane relative to the plane of projection (Fig. 381). The major axis (*3-4*) of the ellipse representing the horizontal projection of the section circle is equal to the diameter of this circle (*3-4* = *1'2'*), the minor axis (*1-2*) being obtained by projecting. Points *5'* and *6'* on the vertical projection of the equator enable us to find the points *5* and *6* at which the ellipse representing the horizontal projection of the circle is separated into visible and invisible parts.

When constructing the projections of a circle obtained as the intersection of a sphere by a plane, use is made of auxiliary planes which cut, for in-

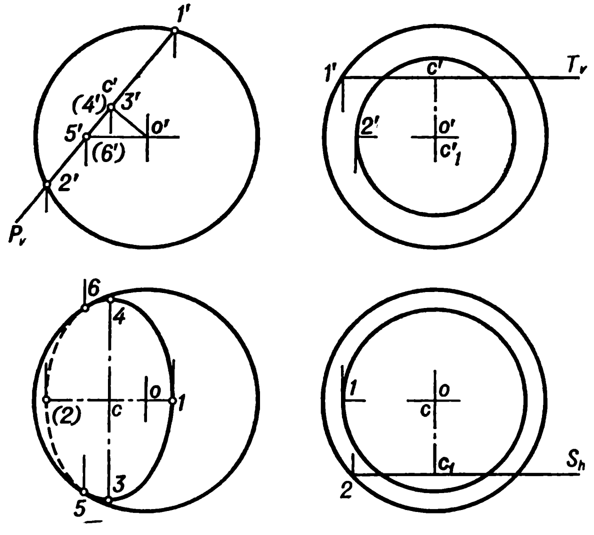

Fig. 381

stance, a sphere in parallels, and a plane in horizontals. We also resort to the transformations of drawings to make the secant plane perpendicular relative to the additional plane of projection.

The curved line of intersection of a torus by a plane is generally constructed also with the aid of planes cutting the torus and the secant plane. For a torus we choose planes cutting it in circles (as we know, the torus has two systems of circular sections: in planes perpendicular to its axis, and in those passing through this axis). In general, the construction scheme is analogous to that shown in Fig. 359. Indeed, as is seen in Fig. 382, auxiliary planes S_1 and S_2 perpendicular to the axis of the torus (in this case an annular torus) cut its surface in circles of radii R_1 and R_2, and the plane P along straight lines projected on the V plane into points $3'$, $5'$, $7'$, i.e. perpendicular to the V plane. In this way the points belonging to the section figure are obtained.

Let us explain the construction represented in Fig. 382. Two representations are given for the annular torus: half of the vertical projection and the profile projection. The torus is cut by a vertical projecting plane P. A semi-circle of radius R_1 is the line of intersection of the torus by an auxiliary vertical plane S_1. This semi-circle touches the trace P_v, thus determining only one point ($3'$, $3''$) belonging to the line of intersection of the torus surface and the plane P and lying in the plane S_1. But if we pass a plane S_2, then it will contain two points belonging to the required line of intersection. The plane S_2 determines on the surface of the torus a semi-circle of radius R_2 which intersects the trace P_v at two points $5'$ and $7'$ representing the vertical projections of the points at which the plane P intersects the torus

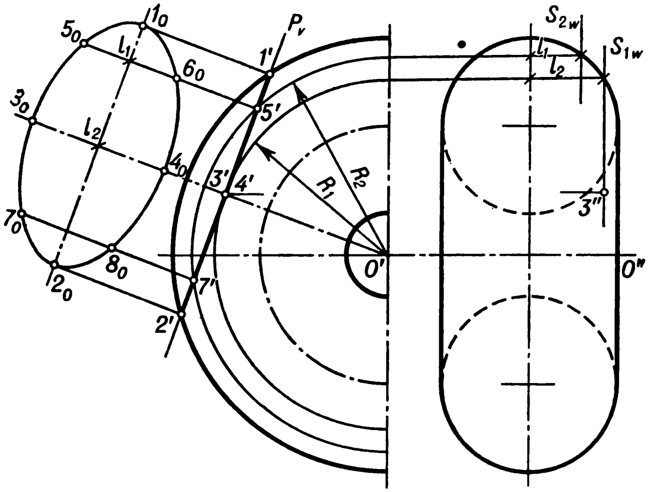

Fig. 382

surface. Proceeding in the same way several times more, we may obtain a number of points for the required line of intersection.

The section figure has axes and centre of symmetry. Determined in the process of construction, the distances l_1 and l_2 of the planes S_1 and S_2 from the vertical plane of symmetry of the torus are used for plotting the points 3_0 and 5_0 when constructing the true size of the section (points 4_0, 6_0, 7_0, 8_0 are plotted making use of symmetry).

The section curve thus obtained resembles the ellipse. But, of course, it is only formal resemblance, since the ellipse is a second-order curve (see Sec. 21), whereas the curved line of intersection of the torus surface is expressed by an algebraic equation of the fourth degree*.

Figure 383 illustrates different sections of an open torus. In the first case it is cut by a plane passing through the axis of the torus ($l = 0$, where l is the distance of the secant plane from this axis) in two circles, in the rest of the cases (2 to 5) in different curves depending on l, R, and r. They are named *Perseus' curves* (in honour of one of the geometers of Ancient Greece). *These are algebraic curves of the fourth degree.*

The curves (2-5) shown in Fig. 383 have different forms: an oval with one axis of symmetry (2), a two-leafed curve with a node at the origin (3), a wavy curve (4), an oval with two axes of symmetry (5) (see Fig. 382). These

*A closed curve constructed in Fig. 382 belongs to *ovals*, i.e. *convex closed plane curves without corner points*. There are ovals made up of circular arcs, and hence constructed with the aid of a compass only. Of course, they are particular cases of ovals.

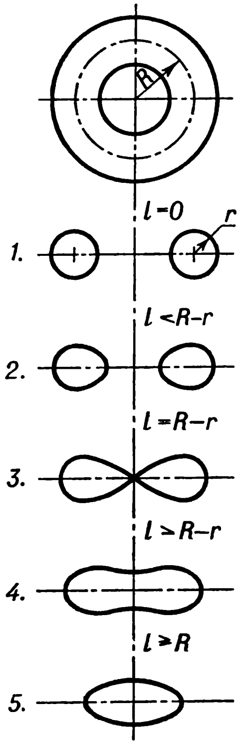

Fig. 383

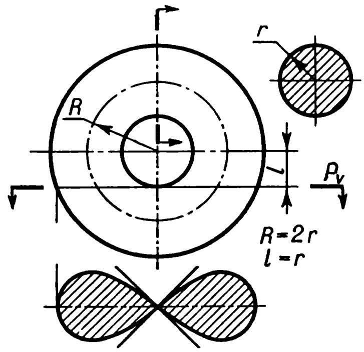

Fig. 384

curves become *ovals of Cassini** (particular cases of Perseus' curves): for an open torus when $R > 2r$, when $R = 2r$, and when $R < 2r$; for a closed ($R = r$) and a self-intersecting ($R < r$) tori if $l = r$, and for an open torus a *lemniscate of Bernoulli*** is obtained (Fig. 384). Its origin is a double point, since the tangents ($y = \pm x$) are mutually perpendicular.

Figure 385 represents a body of revolution whose part under consideration is bounded by three cylindrical surfaces, a conical surface, a spherical surface, and three surfaces of an annular torus, and also by two vertical

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*Jean Dominique Cassini (1625-1712), French astronomer, geographer, and geometer. Oval of Cassini is an algebraic curve of fourth degree which is symmetric about the coordinate axes. It is the locus of points M for which $F_1M \cdot F_2M = a^2$, where F_1 and F_2 are fixed points (foci), and a is a constant.

**James (or Jacques or Jakob) Bernoulli (1654-1705), Swiss physicist, analyst, combinatorist, probabilist, and statistician, the first and perhaps most famous of the Bernoulli family of mathematicians. The lemniscate of Bernoulli is an algebraic curve of the fourth degree having the form of the figure eight. It is the locus of points M for which $F_1M \cdot F_2M = (F_1F_2/2)^2$, where F_1 and F_2 are fixed points (foci).

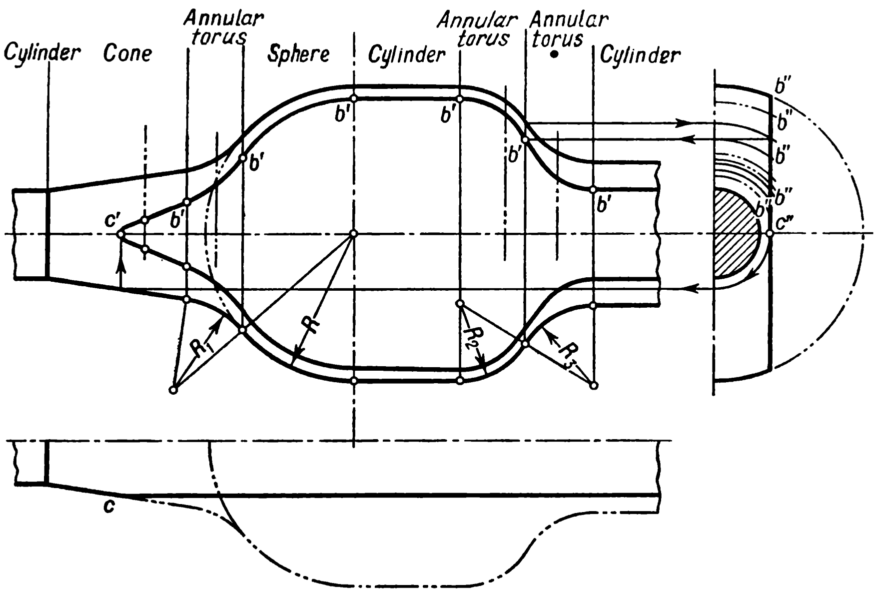

Fig. 385

planes (the drawing gives only half of the top view and half of the profile section view).

Intersecting the surface of the solid of revolution, these planes yield what is called "section lines". Section lines are frequently used in mechanical parts which represent solids of revolution.

First of all we establish "parts" into which the bounding surface is mentally subdivided. This is done with the aid of conjugacy points either on the centre lines, or on the perpendiculars to the elements of the cone and cylinders*. Through the points of conjugacy we pass profile planes cutting each of the surfaces in a circle. The arcs of these circles represented on the W plane determine the profile projection of the reference points on the section line. The position of points b'' determines the position of points b'.

The section line on the cone here is a hyperbola. Its vertex (point c') is found from the obvious position of the projection c''. Knowing the position of the point c'', we determine the projection of the circular arc on which point C must lie. Also shown are "intermediate" points constructed on each part of the section line (one point for each part). The construction is obvious from the drawing.

There is no need to find "intermediate" points on the spherical and cylindrical surfaces, since the sphere is cut in a circle represented on the

*In Figure 385 the conjugacy points are shown only on one half of the front view.

main view without distortion, the radius of this circle being obtained as the greatest of line segments $c''b''$. The cylindrical surfaces are cut along its generating elements.

Sec. 59. Curved Surfaces Cut by a Straight Line

The left-hand drawing of Fig. 386 shows the intersection of a straight line with a cylindrical surface. The surface is specified by its horizontal trace (the curve MN) and the direction of the generatrix (the straight line MT). Through the line AB we pass an auxiliary vertical projecting plane S which intersects the given cylindrical surface along a curve constructed by points at which its elements intersect the plane S. The intersection of the obtained curve with the given straight line AB yields a point K at which the line AB intersects the cylindrical surface.

This is the general method for constructing the points of intersection of a straight line with any surface: *draw an auxiliary plane through the line, find the line of intersection of this plane with the surface; the point of intersection of the given straight line and the line constructed on the surface will be the required point of intersection of a straight line and a surface.*

Here we see a complete analogy with the construction of the point of intersection of a straight line with a plane (see Secs. 22 and 25).

The construction shown in Fig. 386 (left) is, of course, simplified if the auxiliary plane T (the right-hand drawing) is parallel to the element MT: the surface turns out to be intersected along a straight line parallel to MT which is determined by a single point L. This is one of possible particular cases, namely, the given line AB is contained in a plane parallel to the element MT.

Sometimes it is unnecessary to show auxiliary planes. Examples are given in Fig. 387: a right circular cylinder whose axis is perpendicular to the H plane (left), and a right circular cone with the axis in the same position (right). The horizontal projection of the point of intersection of a straight line AB perpendicular to the H plane with the lateral surface of the cone coincides with the horizontal projection of the line itself. Drawing the horizontal projection of the element ST and constructing its vertical projection $s't'$, we find the vertical projection k' of the required point.

An auxiliary plane to be passed through a straight line intersecting a surface should be chosen so that simplest sections are obtained.

For instance, when a conical surface is cut by a straight line, use should be made of a plane passing through its vertex and, hence, intersecting this surface along straight lines. When a cylindrical surface is cut by a straight line an auxiliary plane should be passed through the given line parallel to the generatrix of this surface; then it will intersect the cylindrical surface along straight lines.

Another example with a cone is given in Fig. 388 where the points of intersection are found by means of a plane P defined by the vertex of the cone and the given straight line.

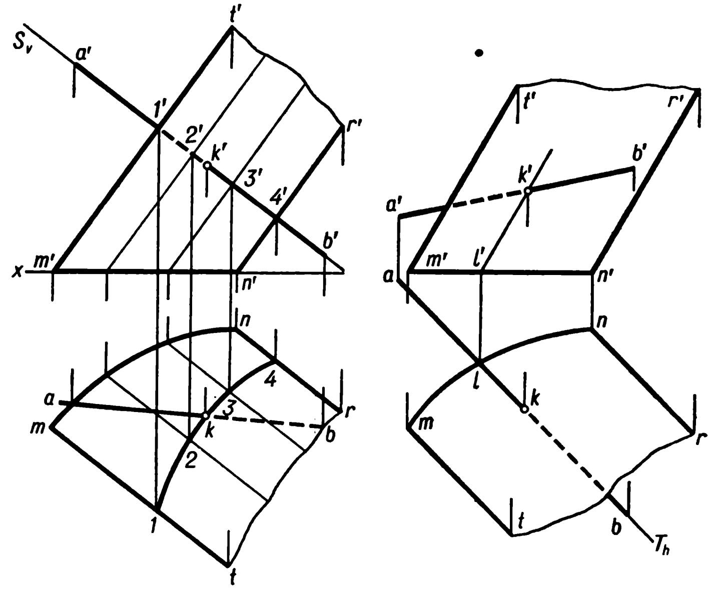

Fig. 386

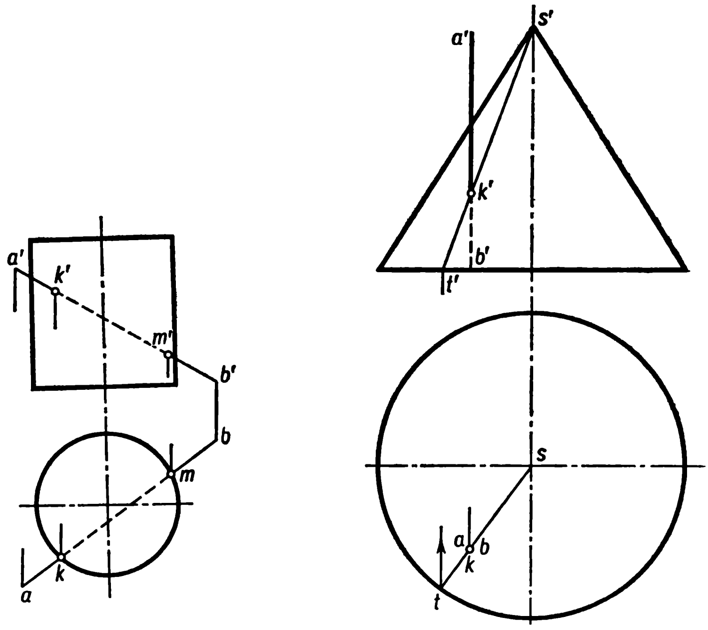

Fig. 387

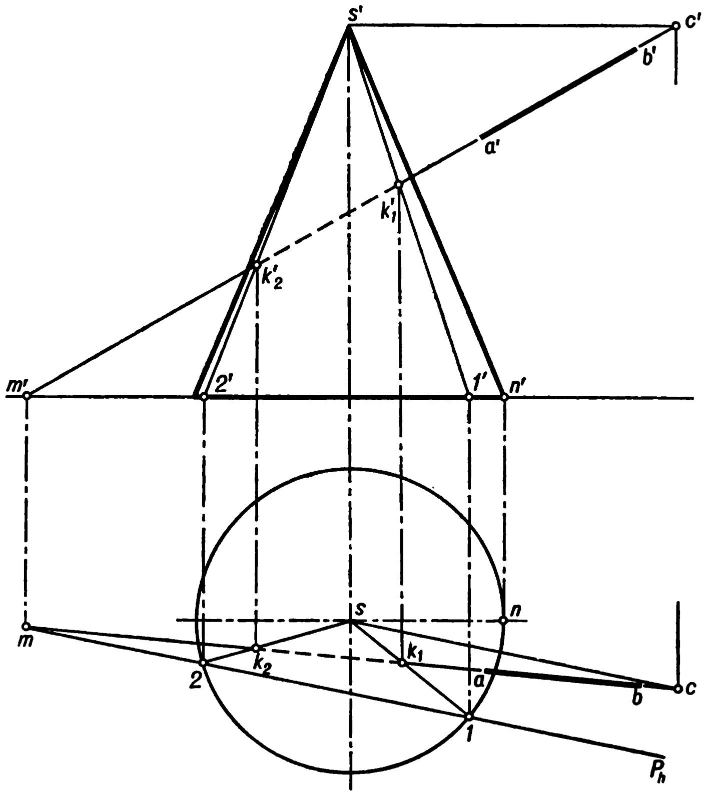

Fig. 388

To construct the elements along which the plane P cuts the cone, we have to find one point more (in addition to the point S) for each element. These points can be obtained as the intersections of the horizontal trace of the plane P with the circumference of the cone base. In Figure 388 the plane containing the base is taken for the horizontal plane of projection (H), therefore the trace of the auxiliary plane is denoted by P_h. For its construction we take an auxiliary straight line SC (a horizontal line contained in the plane P) and find the horizontal trace of the line AB. The trace P_h passes through the point m parallel to the projection sc. The required elements will pass through the points 1, $1'$ and 2, $2'$, the points K_1 and K_2 being the points of entry and exit for the line AB cutting the cone.

If a frustum of a cone is given (Fig. 389) and it is impossible to construct the vertical projection of the vertex, then we may take the point n' for the vertical projection of the point of intersection of the given straight line AM_1 with an auxiliary straight line passing through the vertex S. On finding the

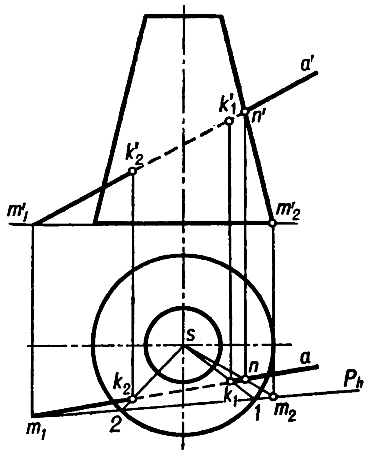

Fig. 389

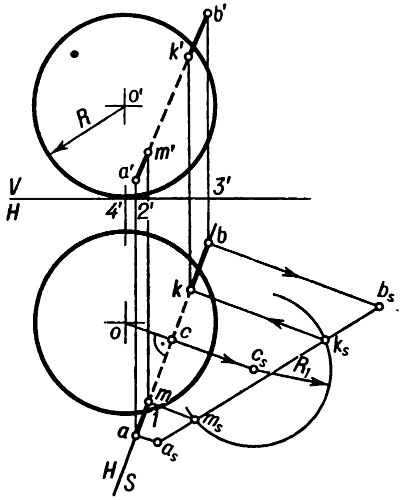

Fig. 390

projection n, we construct the horizontal projection of the auxiliary line SM_2 (using the point s). The further procedure is obvious from the drawing.

Figure 390 illustrates the construction of points K and M at which a line segment AB cuts a sphere of radius R. Here use is made of the method of replacing projection planes.

First of all we pass a horizontal projecting plane S through AB (the horizontal trace of this plane coincides with the projection ab). It cuts the sphere in a circle whose radius R_1 is equal to the line segment $c1$. Taking the same plane S for an additional plane of projection to form a new system S, H, we construct the projection a_sb_s of the line segment AB ($aa_s = a'2'$, $bb_s = b'3'$) and the projection of the circle in which the plane S cuts the sphere. Laying off $c_sc = o'4'$, we find the projection c_s of the centre and from this point as centre strike an arc of radius R_1 to get points k_s and m_s (there is no need to describe a complete circle of radius R_1). Using these points, we first find the projections k and m, and then k' and m'.

One more example of constructing the points of intersection of a straight line with a surface bounding a surface of revolution is given in Fig. 391. Besides two planes, the solid is bounded by two cylindrical surfaces of revolution and an intermediate part belonging to the surface of an annular torus. The straight line intersects one of the cylindrical surfaces at point K_1 and the surface of the annular torus at point K_2. To find the projections of this point, we find the curve with the projection *1-2-3*, *1'2'3'* obtained as a result of intersection of the surface of the torus by the plane S passed through the line AB perpendicular to the H plane. The curve is plotted point by point with the aid of parallels. Two such points (M and N) are shown in the draw-

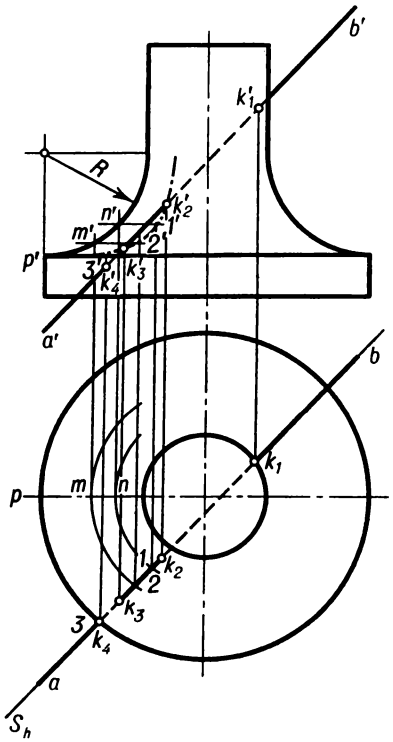

Fig. 391

ing. Then the line pierces the surface of the annular torus at point K_3 and leaves the surface through the point K_4.

Let us now consider the construction shown in Fig. 392 which represents an oblique cylinder with a circular base.

To construct the points of intersection of the cylindrical surface by a straight line *AB*, we pass a plane *P* defined (besides the line *AB*) by an additional straight line BM_1 drawn through the point *B* parallel to the generatrix of the cylinder. Such a plane intersects the cylinder along its elements. If we find the horizontal traces of the straight lines defining the plane, then we can draw the horizontal trace of the plane *P*. Marking the points (*1* and *2*) of intersection of the trace P_h with the cylinder base (contained in the *H* plane), we draw through these points straight lines parallel to the horizontal projection of the generating element of the cylinder and mark the points k_1 and k_2 which are the horizontal projections of the points of intersection of the line *AB* with the surface of the cylinder. Finally, we find the points k_1' and k_2'.

This construction may also be considered as an oblique projection of a cylinder and a straight line *AB* on the horizontal plane of projection. They are projected in the direction parallel to the generatrix of the cylinder. Point *M* of the line *AB* is located in the plane *H*, the point M_1 being an oblique projection of the point *B* on the *H* plane. The line mm_1 is an oblique projection of the line *AB* on the *H* plane. And the cylinder is projected on this plane into its base. The further details are obvious from the drawing.

When solving problems on intersection of a surface by a straight line, it may turn out that the given line does not intersect but only touches the curve bounding the figure in which the given surface is cut by a plane passed through the straight line. In this case the line is tangent to the given surface. In general, if it is required to determine how a line is arranged with respect to a surface, then we have to pass through the line a plane cutting the surface and consider the relative positions of the straight line and the figure obtained as a result of intersection of the surface with a plane.

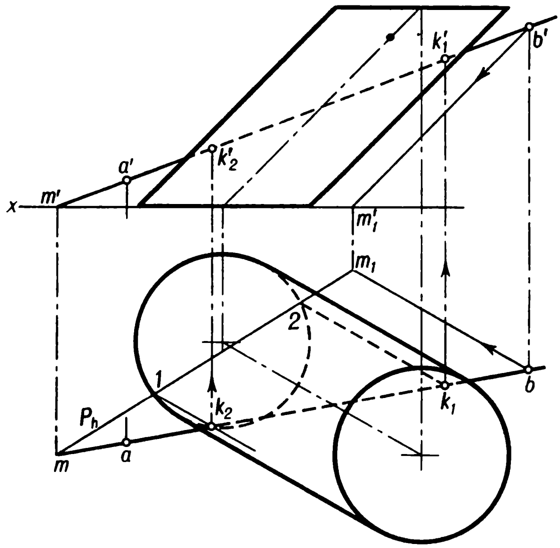

Fig. 392

The present section is dedicated to constructing points obtained as the intersections of a curved surface and a straight line. The common method consists in: (1) passing a plane through a given straight line, (2) constructing the line of intersection of the surface by this plane, (3) finding the points of intersection of the constructed line with the given line.

But how do we proceed if a surface should be intersected not by a straight line but by a plane curve? Obviously, the above described method is applicable in this case as well. Here the plane containing the plane curve is taken for the plane to be passed through the line.

QUESTIONS TO SECS. 58-59

1. In what line is a sphere cut by any plane and what are the projections of this line?

2. What does the method of constructing the section of a torus by a plane consist in?

3. How should the planes cutting a torus in circles be directed?

4. What curves are obtained when a torus is cut by a plane parallel to its axis? In what case do these curves become ovals of Cassini? In what case is the lemniscate of Bernoulli obtained?

5. What is meant under the "section curve"?

6. What does the general method of constructing the points of intersection of a straight line with a curved surface consist of?

7. How should an auxiliary secant plane be passed when a cone is cut by a straight line to obtain straight lines on the surface of the cone?

8. May we use an oblique projection when a straight line cuts a cylinder whose generatrix is not perpendicular to the plane of projection?

CHAPTER 10

THE INTERSECTION OF CURVED SURFACES

Sec. 60. A General Method for the Construction of the Line of Intersection of Surfaces

A general method for the construction of the line of intersection of surfaces *consists in finding the points of this line with the aid of secant planes**. As is clear from the left-hand picture of Fig. 393, the surfaces *I* and *II* are intersected by an auxiliary surface *III* wich intersects the surface *I* along the line *AB* and the sufra(e *II* along the line *CD*. The point *K* at which the lines *AB* and *CD* intersect is common for the sufraces *I* and *II* and, hence, belongs to the line of their intersection. Repreating the foregoing method, we obtain a number of points of the required line. This method was already used when we considered the construction of the line of intersection of two planes (see Sec. 24.). Then the problem was reduced to using two auxiliary planes (see Fig. 166). Eithet of them enabled us to fiind one point common to the planes whose point of intersection had to be found.

Applying the general method for the construction of the line of intersection of two curved surfaces, we may:

(1) cut the surfaces by auxiliary *planes*;

(2) intersect the surfaces by auxiliary *curved surfaces* (for instance, by spheres).

When solving problems, sometimes it is advisable to use auxiliary planes in combination with auxiliary curved surfaces. The auxiliary cutting surfaces should be selected so that the lines of intersection with the given surfaces are as simple as possible and convenient in construction (straight lines or circles).

In general case auxiliary cutting planes are also used for the construction of the line of intersection of a curved surface with a faced surface.

The above described general method for the construction of the line of intersection of two surfaces does not exclude the application of another method, provided one of them is a ruled surface. The latter method consists in the following: find the point at which the rectilinear generatrix of one

*This line is often called a transition line.

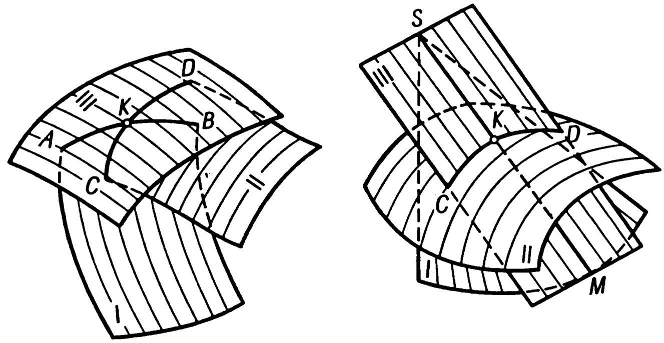

Fig. 393

surface intersects the other surface, and, repeating this construction for a number of elements, draw the required line through the found points. The right-hand picture of Fig. 393 illustrates how through the element SM of the surface I a plane III is passed which cuts the second surface (II) in a curve CD; the element SM intersects this curve at point K through which the required line of intersection of the surfaces I and II will pass.

This is also applicable to the case when a curved surface is intersected with a faced surface: here the role of elements is played by the edges of the faced surface.

Thus, *to construct the points of the lines obtained on one surface when it is cut by another surface, we use all kinds of auxiliary cutting planes (oblique planes included), curved surfaces, rectilinear generating elements of curved ruled surfaces, and edges of faced surfaces. If necessary, we resort to the methods for transformation of the drawing in order to simplify and specify the constructions involved.*

In most examples given below we consider geometric solids, i.e. portions of space bounded with various surfaces. Of two surfaces only one intersects the other. Therefore, one of the surfaces is preserved forming holes in the intersecting surface. We usually distinguish between (1), *penetration* with two separate lines of intersection (see, for example, Fig. 412, where a cone with a horizontal axis enters another cone), or one line with a node (Fig. 427), and (2) *cutting* when one line is obtained (see, for example, Figs. 396 and 426).

Cast parts usually have smooth transitions, i.e. transitions from one surface to another via an intermediate surface (say, a torus). Then, to indicate a transition, we construct the line of intersection (transition line) of geometric forms which are the basis of technical forms (see, for instance, Fig. 399 and 430)*.

*In similar cases, i.e. when a monolithic solid is considered, it is more precise to speak of the line of coupling the surfaces.

The projections of the line of intersection are obtained within the limits of the common portion of the projections of both surfaces.

In constructing the projections of a transition line first find so-called obvious points which are determined without any constructions. Then determine the reference (definitive) points located, for instance, on the extreme generating elements of surfaces of revolution or the extreme edges separating the visible portion of the transition line from the invisible. Classified under reference points are also the extreme points of an intersection line: the uppermost and lowest relative to the H plane, nearest and farthest with respect to the viewer, right-hand, left-hand, and so on.

Sec. 61. The Choice of Auxiliary Cutting Planes in the Cases when They Can Intersect Both Surfaces Along Straight Lines

When both surfaces are cylindrical or conical, or one of them is cylindrical and the other is conical, in a number of cases axiliary planes should be chosen so that they intersect both surfaces along straight lines (generating elements of these surfaces). The point of intersection of an element of one surface with an element of the other belongs to the line of intersection.

Figure 394 gives an example of choice of cutting planes for the case of mutual intersection of two cylinders, the plane P ("plane director") defined by two intersecting lines LM and LN (respectively parallel to the generatrices of the cylinders) serving as "pattern" for them. This is an oblique plane; consequently, in the present case the auxiliary secant planes are also oblique. It is sufficient to take the horizontal traces of such planes, drawing them parallel to the trace P_h, since the directions of the straight lines along which these planes intersect both cylinders are known—they are parallel to their directrices. For instance, the trace P_{1h} parallel to P_h intersects either of the directrices of the given cylinders at two points which enables us to determine their generatrices. These generatrices intersect at four points which just belong to the required line of intersection. The construction is carried out proceeding from the assumption that one cylinder penetrates into the other making two holes in its surface.

Obviously, in such construction we may take a random element of one cylinder, draw the trace of an auxiliary plane through the trace of this element, as it is done with the trace P_{1h}, and examine whether this plane yields points of intersection with the elements of the other cylinder obtained with the aid of the same plane.

The pattern of auxiliary secant planes for the cases when a cylinder penetrates into a prism or vice versa is constructed in a similar way.

Figure 395 depicts the line of intersection constructed for the case when a pyramid penetrates into the surface of a cylinder. To choose the planes which would intersect along straight lines not only the faces of the pyramid but also the cylindrical surface (along its elements), a straight line SM is drawn parallel to the generatrix of this surface and passing through the vertex

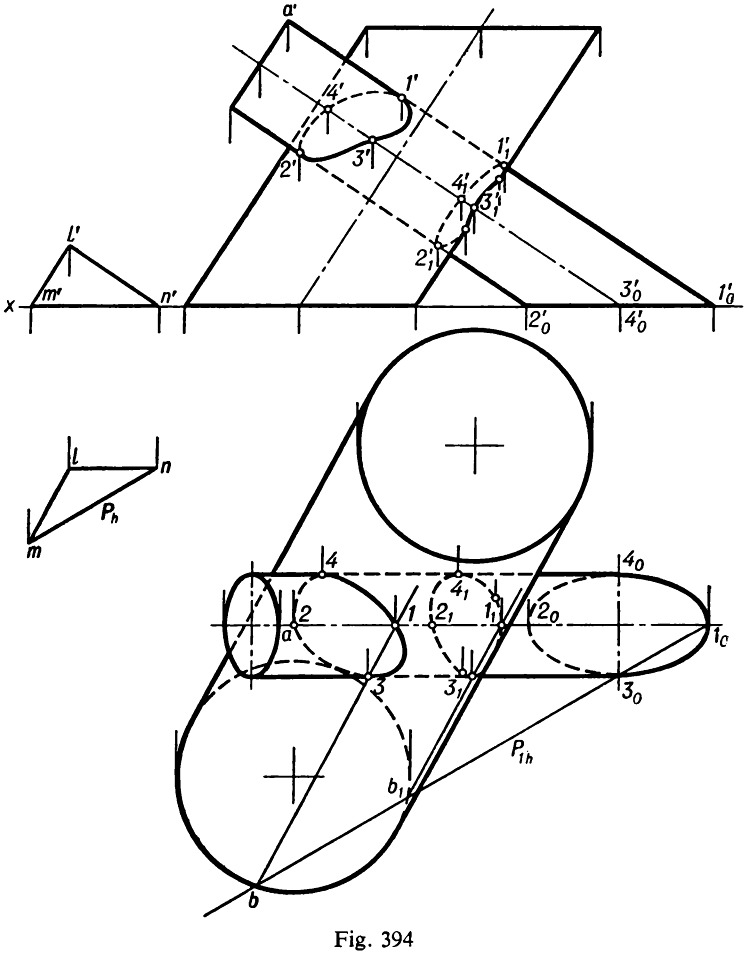

Fig. 394

of the pyramid. Obviously, if instead of a pyramid a cone is taken, then we proceed in the same way: we draw a straight line through the vertex of the cone parallel to the generatrix of the cylindrical surface. The horizontal traces of the auxiliary cutting planes must pass through the point *m* which will correspond to passing the planes through the line *SM*. The horizontal traces of the planes intersect the horizontal traces of the lateral surfaces of the cylinder and pyramid at points through which the horizontal projections of the lines of intersection of the auxiliary planes with the given surfaces pass. For instance, the trace T_h intersects the horizontal projections of the sides of the pyramid base at points *d* and *e* which corresponds to the intersection of the faces *SBC* and *SAC* by the plane *T* along the straight lines *SD*

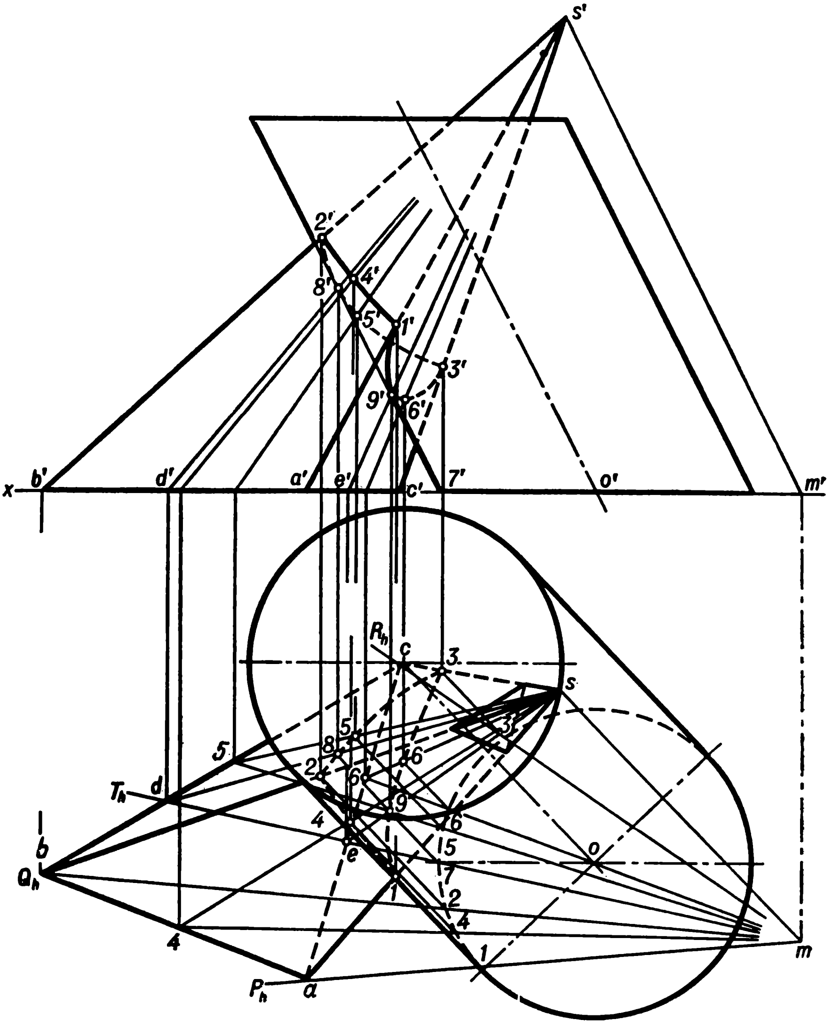

Fig. 395

and *SE*. But the same plane *T* cuts the cylindrical surface along the element with the initial point *7*, *7'*. The intersection of this element with the lines *SD* and *SE* yields points *8*, *8'* and *9*, *9'* belonging to the line of intersection. This line is located on the cylindrical surface, since in this case the pyramid penetrates into the cylinder piercing its upper base to make a triangular hole in it.

The curves obtained on the given cylindrical surface are elliptic arcs, since they represent the intersections of this surface by planes (faces of the

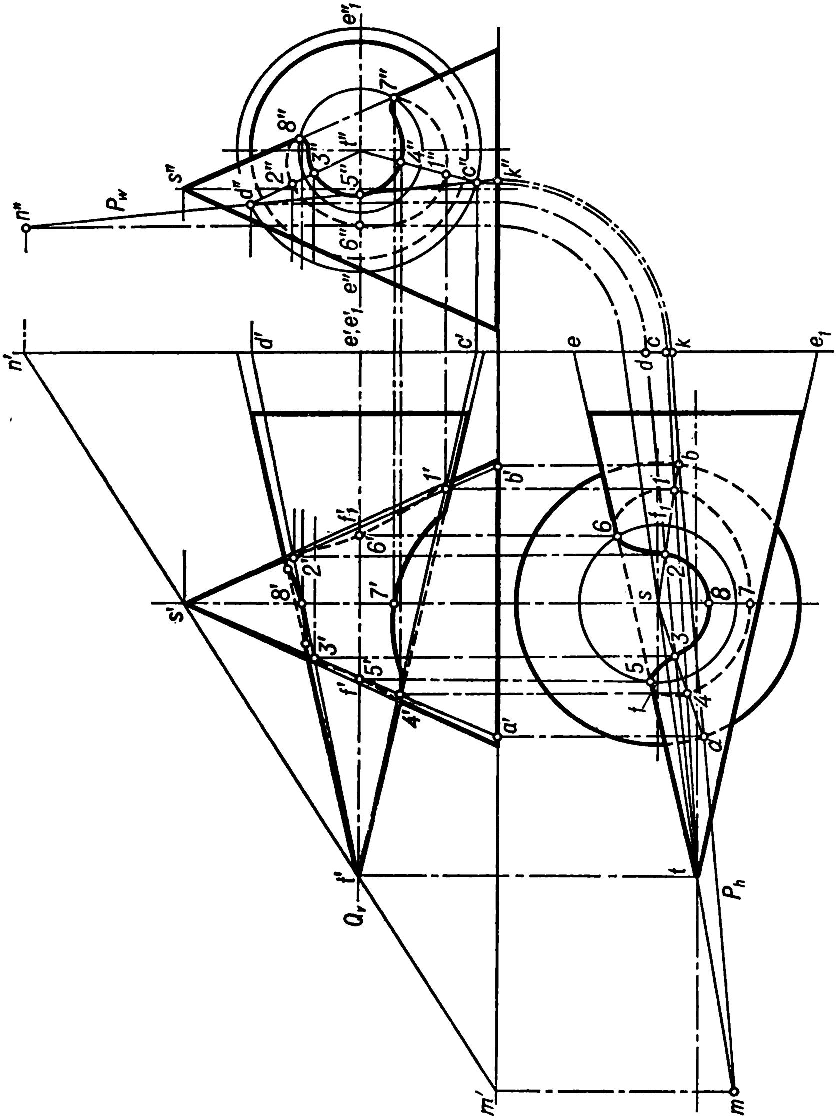
n''
s''
P_w
d''
2''
3''
8''
5''
6''
1''
4''
7''
c''
k''
e''
e''_1
$e';e'_1$
n'
d'
c'
s'
8'
2'
3'
f'
5'
4'
6'
f'_1
7'
1'
a'
b'
f''
Q_v
m'
m
t
P_h
e
e_1
d
c
k
b
1
f_1
6
2
s
8
7
3
5
f
4
a

Fig. 396

pyramid). It is necessary to begin the construction with finding the points of intersection of the edges of the pyramid with the cylinder.

Figure 396 demonstrates the line of intersection formed on the surface of a cone (with the vertex *S*) when the latter is cut by a cone with the vertex *T*.

In this case the points belonging to the line of intersection are determined by oblique planes each of which must pass through the vertices of both cones.

Prior to all relevant constructions, a straight line is drawn through the vertices *S* and *T*. The planes passing through the line *ST* cut the conical surfaces along their elements.

These planes form a pencil with the line *ST* as its axis. Constructing the horizontal trace of this line, we get a point *m* through which the horizontal traces of the required planes must pass, for instance, the trace P_h. Intersecting the circumference of the base circle of the cone with the vertex *S*, the trace P_h yields points *a* and *b* with the aid of which we can find the horizontal projections of the elements *SA* and *SB* on the surface of this cone. Then we find the vertical projections of these elements ($s'a'$ and $s'b'$).

We came across a similar method in Fig. 282 representing a mutual intersection of two pyramids.

But in this case the horizontal trace P_h does not enable us to determine the elements of the cone with the vertex *T* lying in the plane *P*. Therefore we find the profile trace P_w which cuts the line of intersection of the conical surface with the *W* plane at points c'' and d''. On constructing the horizontal and vertical projections of the points *C* and *D*, we draw the elements of the cone with the vertex *T*: *CT* and *DT* ($c't'$, ct and $d't'$, dt). The found elements intersect at points belonging to the required line.

Passing a number of auxiliary planes through *ST*, we can construct a number of points belonging to the required line of intersection and pass through them a smooth curve.

Comparing the constructions represented in Fig. 396 with those of Figs. 394 and 395, we see that in them it was sufficient to construct only the horizontal traces, and in the case represented in Fig. 396 the profile traces were also needed. This is explained by that the bases of the solids considered in Figs. 394 and 395 are located on the *H* plane, while in Fig. 396 only one of the cones rests against the *H* plane. Therefore, when the bases of the solids are contained in different planes of projection (Fig. 397), we have to use the corresponding traces of the cutting planes. But if, as in Fig. 396, the surface of one of the cones does not meet the plane of projection, then it is extended to this plane by constructing its trace.

The passing of cutting planes through the line joining the vertices of the cones is obviously suitable for the case when a conic surface is intersected by a pyramid.

In Figure 396 not only oblique planes (for instance, plane *P*) but also planes of particular positions are used to find some points. Thus, the plane passed through the point *T* parallel to the *H* plane (the trace Q_v) cuts the

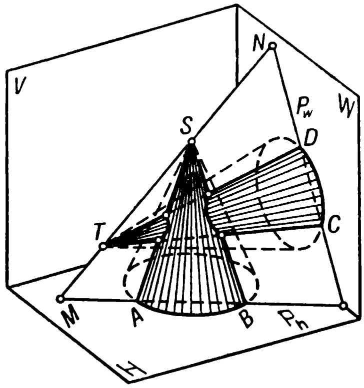

Fig. 397

cone along the elements TE and TE_1, and the cone with the vertex S in a circle FF_1. The intersection of its horizontal projection with et yields the horizontal projections 5 and 6 by which we find the vertical and profile projections $5'$, $6'$ and $5''$, $6''$. Passing through S a profile plane, we determine the points with the projections 7, $7'$, $7''$ and 8, $8'$, $8''$.

Sec. 62. The Use of Auxiliary Cutting Planes Parallel to the Projection Planes

The use of auxiliary cutting planes was already demonstrated in Fig. 396 where one plane was parallel to the H plane, and the other to the W plane. But the leading role was played by a pencil of oblique planes with a common straight line ST. We are going here to consider some examples of finding the points belonging to the required curve only with the aid of planes parallel to the projection planes. It happens when such planes cut the intersected surfaces in straight lines or in circles.

In Figure 398 a frustum of a cone whose axis is perpendicular to the W plane penetrates into a hemi-sphere forming a closed curve on its surface. In this case the points on the line of intersection are found with the aid of planes parallel to the W plane and perpendicular to the axis of the cone. The planes P and P_1 cut the surface of the hemi-sphere in circles of radii $o'a'$ and $o_1'a_1'$ and the conical surface in circles of radii $c''b''$ and $c''b_1''$. On constructing these circles on the W plane, we find the profile projections of the points belonging to the required line. Thus, at the intersection of the circles obtained with the aid of the P plane we mark points $1''$ and $2''$. The vertical and horizontal projections of these points lie on the traces P_v and P_h. Using the plane P_1, we find the points 3, $3'$ and 4, $4'$.

Since the axis of the cone is parallel to the H plane, passing through it a plane Q parallel to the H plane, we cut the conical surface along its elements,

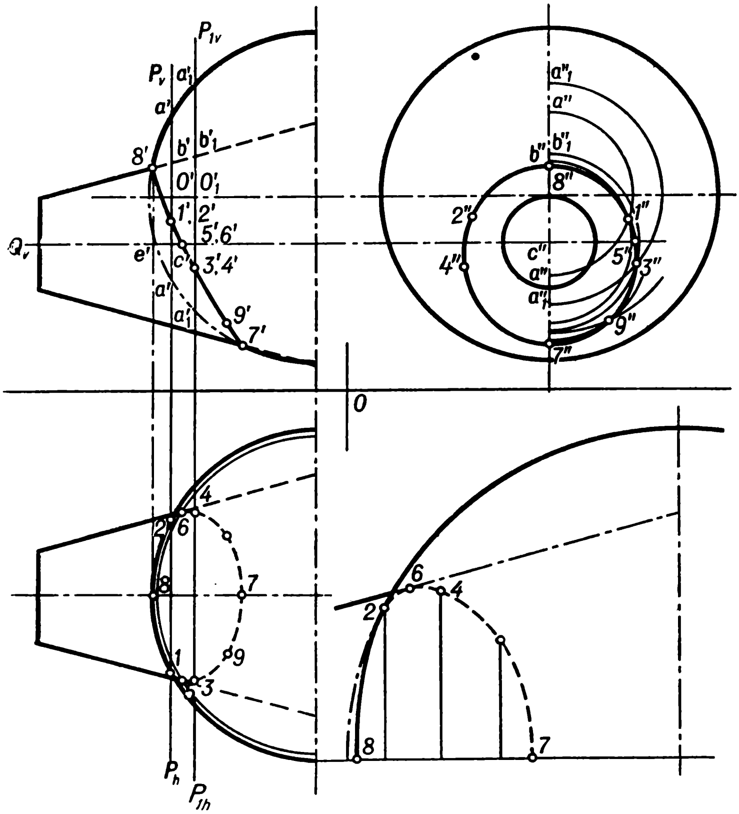

Fig. 398

and the spherical surface in a circle. Constructing the projection of the circle on the *H* plane, we find the points *5* and *6* as the intersections with the projections of the corresponding elements of the cone.

In this example the positions of the points *7*, *7'* and *8*, *8'* are obvious. These points, as also the points *5*, *5'* and *6*, *6'*, belong to reference points; aen nearged drawing shows the construction of point *6* in which the projections of the element of the cone and the curve of intersection touch each other.

Another example is given in Fig. 399 where the points of the line of intersection of two surfaces are found with the aid of cutting planes parallel to the *H* plane and in one case (point *B*) to the *W* plane. Here it is more appropriate to speak of a transition line, since the represented part (a bearing cap) is manufactured by casting and where the conical surface merges with the spherical no pronounced line of intersection is obtained. But constructed in Fig. 399 is just the line of intersection, since we consider geometrical forms here.

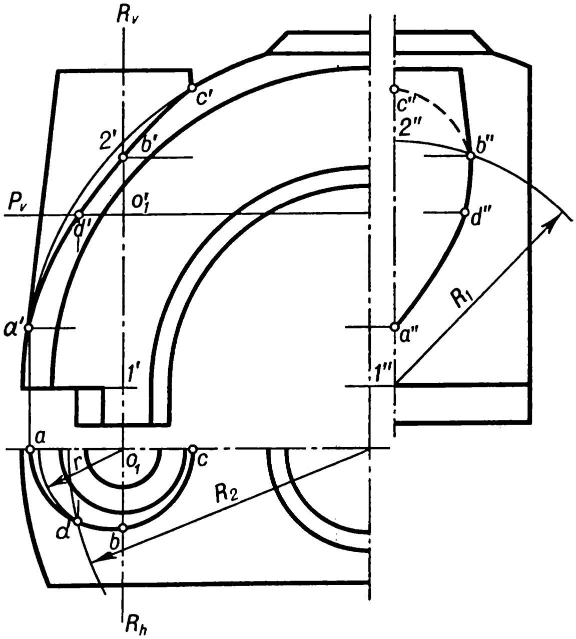

Fig. 399

The procedure is clear from the drawing. To construct the projections of the point B which is important for determining the transition between the projections of the cone element and the line of intersection on the W plane (point b''), a profile plane passing through the cone axis is taken. The spherical surface is cut in a circle of radius $R_1 = 1'2'$. First we find the projection b'', and then b' and b. The point B, the same as A and C, is a reference point*.

QUESTIONS TO SECS. 60-62

1. What does the general method for constructing the line of intersection of two surfaces consist in?

2. If at least one of the intersecting curved surfaces is a ruled one, is it possible to construct the line of intersection using the points of intersection of the elements of this ruled surface with the other surface?

3. What is the difference between "penetration" and "cutting" occurring when one surface intersects the other?

*For the projections of the line of intersection of a spherical and conical surfaces see Sec. 65.

4. Within the limits of what part of the projections of intersecting surfaces is the projection of the line of intersection obtained?

5. What points of the line of intersection of surfaces are called reference points?

6. What recommendations for choosing auxiliary cutting planes may be made for the cases of intersection of cylinders, cones, prisms, and pyramids?

7. In what cases is it recommended to use auxiliary secant planes parallel to the projection planes for constructing the line of intersection of two surfaces?

Sec. 63. Some Special Cases of Mutual Intersection of Two Surfaces

1. Figure 400 represents the intersections of: (1) two cylinders with parallel generatrices, (2) two cones with a common vertex. In both cases the lines of intersection of the surfaces are generating elements common to these surfaces.

Suppose it is required to construct the projections of the straight line passing through the point B on the axis of projection and inclined at an angle α to the H plane and at an angle β to the V plane. It is known that for an oblique line $\alpha+\beta < 90°$ (see Sec. 13).

The locus of straight lines passing through a given point and making an angle α with the H plane is a conical surface of revolution whose vertex is situated at the given point and the generatrix forms an angle α with the H plane.

Analogously, the locus of straight lines passing through a given point and forming an angle β with the V plane is a conical surface of revolution whose vertex is situated at the given point, and the generatrix forms an angle β with the V plane.

Obviously, the required straight line must simultaneously belong to both conical surfaces having a common vertex at the given point, i.e. must be the line of their intersection, or in other words, their common generating element. We get eight rays emanating from point B which meet the preset conditions (four straight lines).

Figure 401 shows how one of these rays is constructed. The first cone is determined by the generatrix BA_1 and the axis perpendicular to the H plane; the second cone is defined by the generatrix BA_2 and the axis perpendicular to the V plane. For the present we have only one point B (a common vertex of the cones) for constructing the required straight line. The second point (point K) which is common for the surfaces of these cones is found with the aid of sphere with point B as centre (see Fig. 415).

Another example when in the process of some construction we use the property of the intersection of two conical surfaces with a common vertex along a common straight line (element) is the construction of the element of a

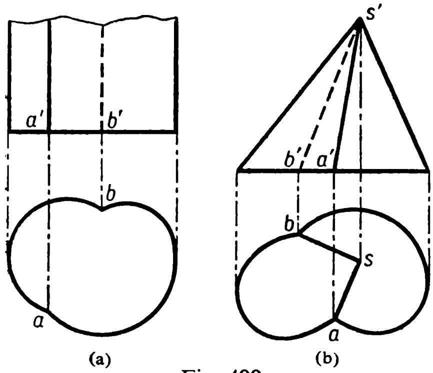

Fig. 400

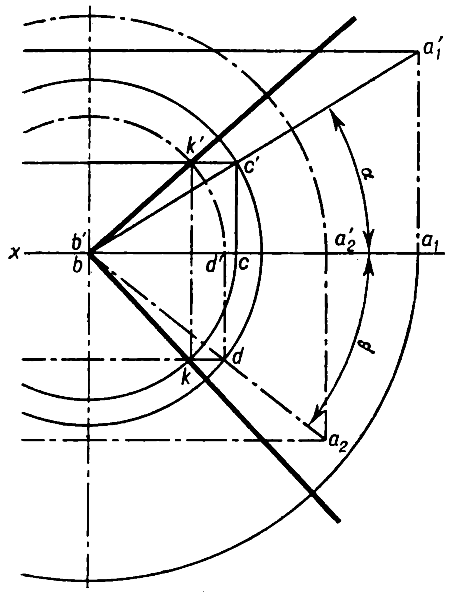

Fig. 401

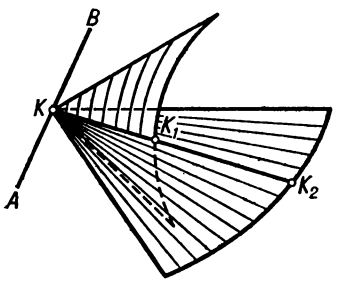

Fig. 402

ruled surface called a cylinder with three directrices (for this surface see Sec. 50, B, Item 2.2). Suppose we are given a straight line AB and curved lines as the directrices (Fig. 402). If we take a point (K) on the rectilinear directrix as a common vertex of auxiliary conical surfaces for which the given curves serve as the directrices, then the straight line of intersection of these conical surfaces, passing through their vertex, will also intersect their directrices, i.e. will turn out to be a rectilinear generatrix of the cylinder with three directrices. Obviously, we have to take a number of points of the given straight line and to carry out the above mentioned construction for each of them which will yield several elements of the cylinder with three directrices.

If the three directrices of this surface are curved lines, then the indicated method of construction remains the same: the points serving as vertices for auxiliary conical surfaces are taken on one of the given curves.

2. In some cases of intersection of surfaces of revolution of the second order the line of intersection decomposes into two plane curves of the second order. It happens when both intersecting surfaces of revolution (a cylinder and a cone, two cones, an ellipsoid and a cone, etc.) are circumscribed about a common sphere. In the examples given in Fig. 403 the lines of intersection in the first three cases are ellipses, in the fourth case an ellipse and a parabola, and in the fifth case an ellipse and a hyperbola.

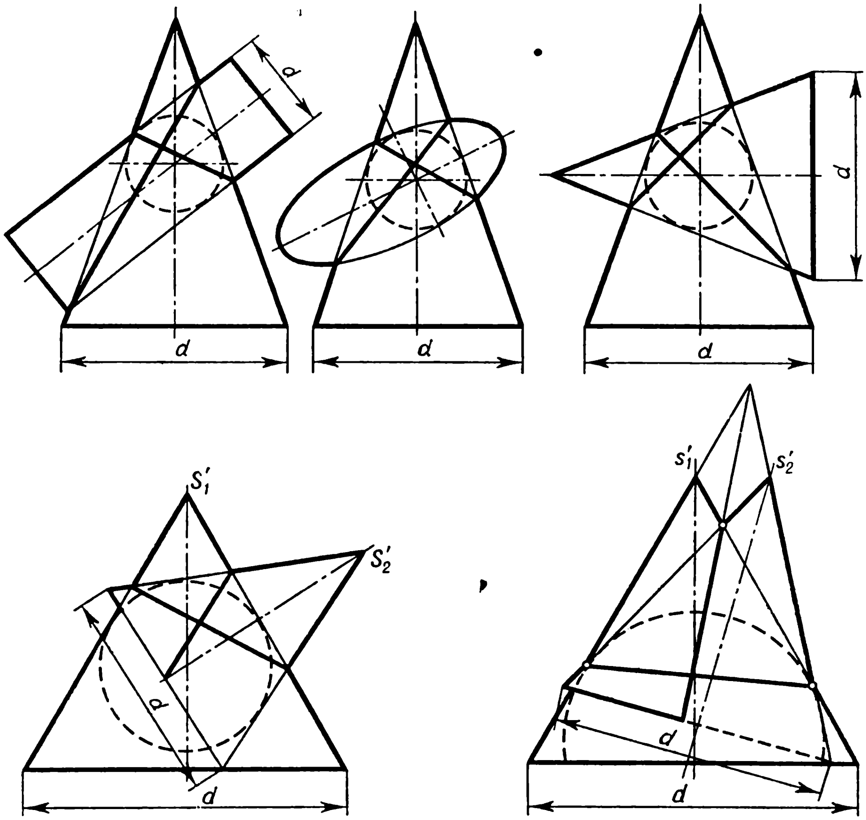

Fig. 403

Figure 404 (left) represents two cylinders of equal diameters with intersecting axes. From the point of intersection of the axes a sphere can be drawn inscribed in both cylinders. The surfaces intersect along a line consisting of two ellipses. The right-hand drawing of the same figure also represents two cylinders of equal diameters, but this time their axes intersect not at right angles. The line of intersection is made up of two semi-ellipses.

The curved lines of intersection of the surface depicted in Figs. 403 and 404 are projected on the vertical plane of projection into rectilinear segments, since a common plane of symmetry for each pair of the considered surfaces is parallel to the V plane.

In the examples under consideration we come across a *double contact* of two intersecting surfaces of second order, i.e. the surfaces have two points of tangency, and, consequently, two planes either of which touches both surfaces at their common point. The above mentioned constructions are

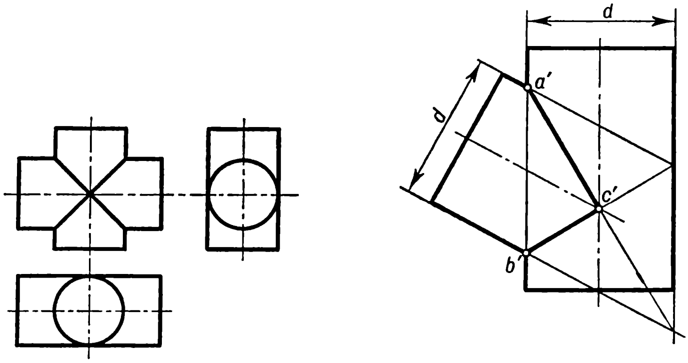

Fig. 404

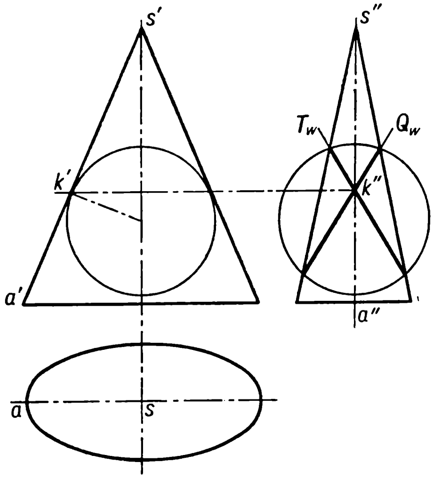

Fig. 405

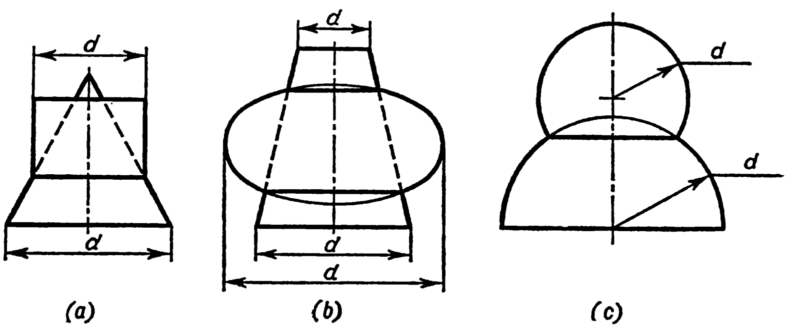

Fig. 406

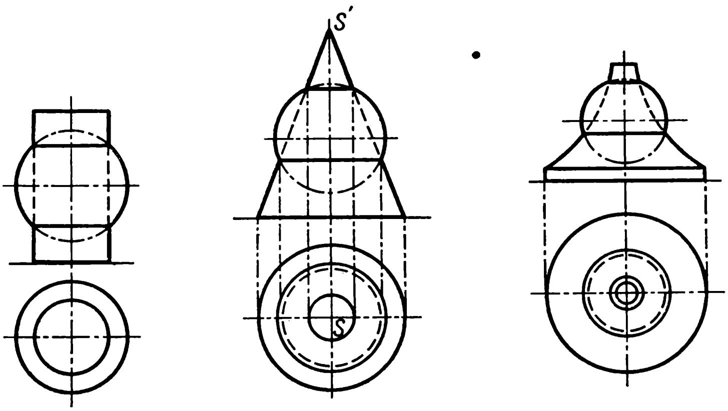

Fig. 407

based on the following two statements:* (1) *second-order surfaces having a double contact intersect each other along two second-order curves, the planes of these curves passing through a straight line defined by two points of contact;* (2) *two second-order surfaces circumscribed about a third second-order surface (or inscribed in it**) intersect each other along two second-order curves.* The second statement known as *Monge's theorem* follows from the first one.

On the basis of these statements we can find the circular sections of an elliptic cone and an elliptic cylinder (see Sec. 50). An example is given in Fig. 405 where a sphere is taken so that it has a double contact with the surface of an elliptic cone. The sphere and cone intersect each other along two plane curves which are circles contained in profile projecting planes T and Q represented in the drawing by their profile traces. Planes parallel to the planes T and Q produce two systems of circular sections of the elliptic cone.

3. Coaxial surfaces of revolution (i.e. surfaces with a common axis) intersect along circles. Figure 406 gives three examples: (a) a cylinder and a cone, (b) an oblate ellipsoid and a frustum of a cone, (c) two spheres. In all these examples only the vertical projections are given, a common axis of the surface being arranged parallel to the V plane. Therefore the circles obtained as the intersections of the surfaces are projected on the V plane into rectilinear segments.

Any diameter of a sphere may be taken for its axis. Therefore intersecting spheres are considered as coaxial surfaces of revolution. A cylinder and a sphere, a cone and a sphere, a surface of revolution and a sphere represented in Fig. 407 may also be considered as coaxial surfaces. The axes of the

*For their proofs see a course of analytic geometry.
**For instance, two oblate ellipsoids of revolution inscribed in a spherical surface.

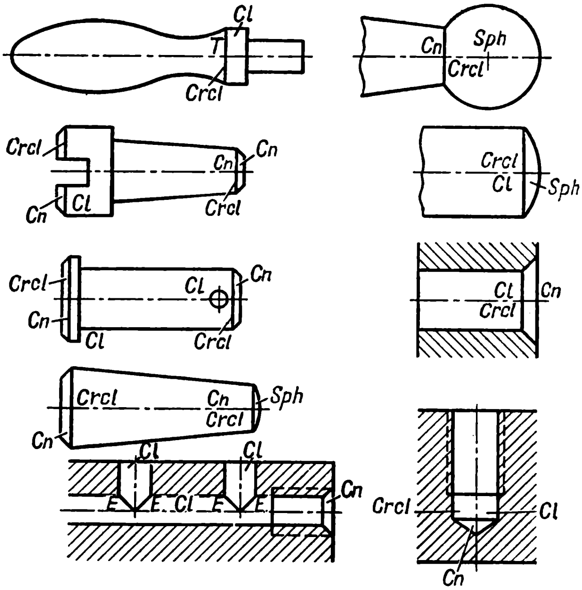

Fig. 408

cylinder, cone, and surface of revolution pass through the centres of the spheres. Their intersections are circles.

Figure 408 gives examples of representation of coaxial surfaces of revolution and intersecting drilled holes of the same diameter used in machine drawing. The surfaces are designated by letters in the following ways: annular torus by *T*, cone by *Cn*, cylinder by *Cl*, sphere by *Sph*. The lines of intersection are also denoted by letters: a circle by *Crcl*, an ellipse by *E*. These lines are projected into rectilinear segments, since the axes of the represented surfaces are parallel to a projection plane (in the present case to the *V* plane).

Sec. 64. The Use of Auxiliary Cutting Spheres

The intersection of surfaces of revolution with a sphere considered in the preceding section underlies the use of spheres as auxiliary surfaces in the construction of the lines of intersection of the given surfaces.

Given in Fig. 409 are two surfaces of revolution with intersecting axes and, consequently, with a common plane of symmetry parallel to the *V* plane. From the point of intersection of the axes we can describe a number of spheres. Suppose a sphere designated as *Sph. 1* is constructed. It cuts either of the surfaces in circles whose intersections yield the points common for both surfaces and, hence, belonging to the line of intersection. As is

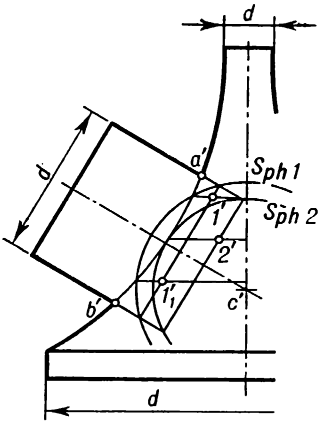

Fig. 409

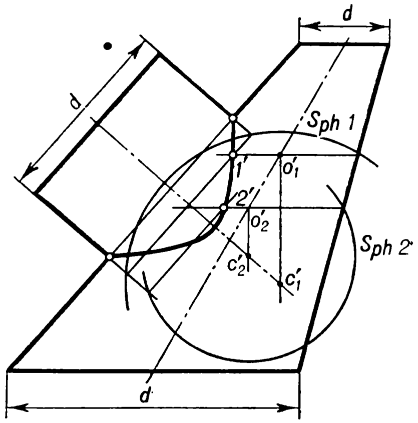

Fig. 410

seen from the drawing, the construction is considerably simplified due to the fact that the plane of symmetry common for the given surfaces is parallel to the *V* plane. Therefore, the circles in which a sphere intersects both surfaces is projected on the *V* plane into rectilinear segments. Moreover, the projection of the line of intersection is constructed without using the other projections of the surfaces.

Of course, we construct several spheres in order to get a sufficient number of points necessary for drawing the required projection of the line of intersection. Figure 409 shows one more sphere *(Sph. 2)* which only touches the surface with a curved-line generatrix yielding point *2'*. This point is a "last" one for the vertical projection, since spheres of smaller diameters will not supply points for the required line.

Finally, we have to draw through the points a', $1'$, $2'$, $1'_1$ and b' a smooth curve which will be the vertical projection of the line along which the two surfaces are joined into a single unit.

Thus, the whole construction is carried out using only the front view.

So, if it is required to construct the line of intersection of two surfaces of revolution with intersecting axes, then we may use auxiliary cutting spheres with the centre at the point of intersection of the axes of the surfaces.

Figure 410 supplies another example illustrating the use of spheres in a construction analogous to that shown in Fig. 409. This time only one of the intersecting surfaces is a surface of revolution, the other being an oblique circular cone (see Sec. 50). The latter has a number of circular sections parallel to one another.

Each of these sections may be taken for a parallel of the sphere whose centre is taken on the axis of the cylindrical surface. For instance, taking a parallel with centre O_1 (the projection o'_1), we draw through O_1 a per-

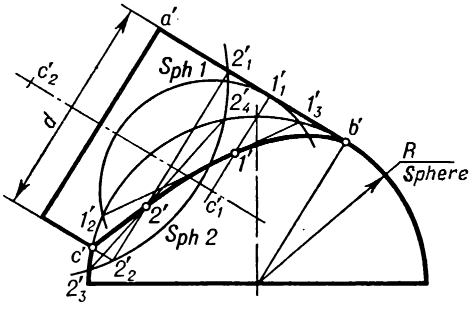

Fig. 411

pendicular to the plane containing the parallel to intersect the axis of the cylinder. The point C_1 (projection c'_1) is taken for the centre of the sphere cutting either of the surfaces in circles: the conical surface is cut along the taken parallel with centre O_1, and the cylindrical surface in a circle obtained by "pulling" the surface onto the sphere. As a result, on the vertical projection we get the point *1'* belonging to the projection of the required line of intersection. Analogously, we can find the centre C_2 (projection c'_2) for constructing a sphere with the aid of the chosen parallel whose centre is at point O_2 (the projection $\bar{o}'_2$). The further constructions are clear from the drawing.

Thus, *auxiliary spheres may also be used in the cases when a surface of revolution is intersected by a surface having parallel circular sections whose centres lie on a single line intersecting the axis of the surface of revolution.*

Figure 411 illustrates the construction of the line of intersection (for rather "coupling") of a surface of revolution and a sphere (the element *AB* of the cylinder touches the sphere at point *B*). The surfaces have a common plane of symmetry parallel to the *V* plane. The centre of one auxiliary sphere *(Sph. 1)* is taken at the point with the vertical projection c'_1. The radius of this sphere is taken equal to the line segment $c'_1 1'_1$ (in this case this is the least radius for an auxiliary sphere). At the same time it is the radius of the circle along which the auxiliary sphere *Sph. 1* touches the cylindrical surface. This sphere cuts the given sphere of radius *R* in a circle of diameter $1'_2 1'_3$. The intersection of the straight lines $1'_2 1'_3$ and $c'_1 1'_1$ yields point *1'* which is one of the points belonging to the projection of the required line of coupling of a cylinder and sphere.

The second auxiliary sphere *(Sph. 2)* is drawn from the point also taken on the axis of the cylinder (its projection is c'_2). This sphere yields the point *2'*.

On obtaining some more points between the extreme points *b'* and *c'*, we can draw the vertical projection of the required line. At point *1'* obtained with the aid of the "limiting" sphere (inscribed in the cylinder) the straight line $1'_2 1'_3$ is tangent to the curve *b'1'2'c'*.

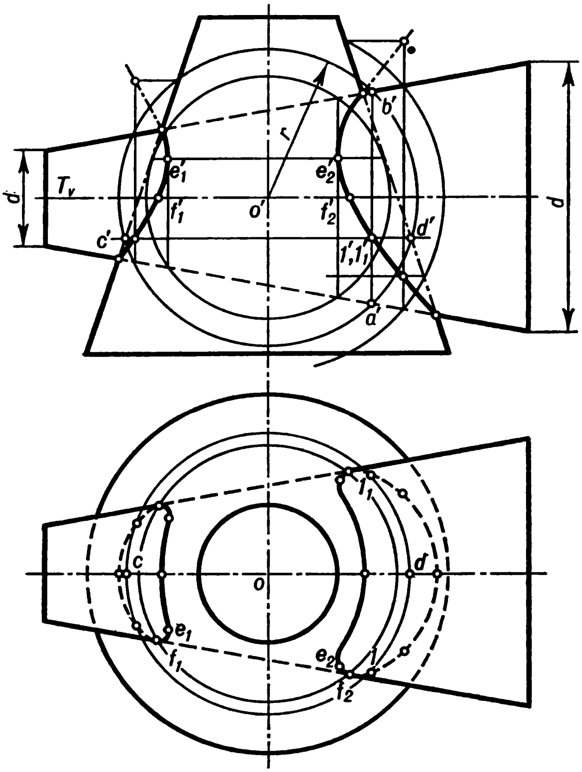

Fig. 412

Figure 412 shows the intersection of two cones of revolution. The intersection of their axes forms a common (for these cones) plane of symmetry parallel to the *V* plane.

In this case we use auxiliary spheres described from one and the same centre located at the point *O* where the axis of the cones intersect. For instance, to find the point *1*, a sphere of radius *r* is constructed.

The points e'_1 and e'_2 on the vertical projection which are the nearest to the cone with a vertical axis are determined with the aid of a sphere inscribed in this cone*.

The points f_1 and f_2 which separate the horizontal projection into visible and invisible portions are determined with the aid of a plane *T* passing

*The line of intersection of two second-order surfaces having a common plane of symmetry is projected on a plane parallel to the plane of symmetry into a second-order curve. In the present case a hyperbola is obtained with the points e'_1 and e'_2 as its vertices. In Figure 411 the vertical projection of the line of coupling of the surfaces is a parabola (see Sec. 65).

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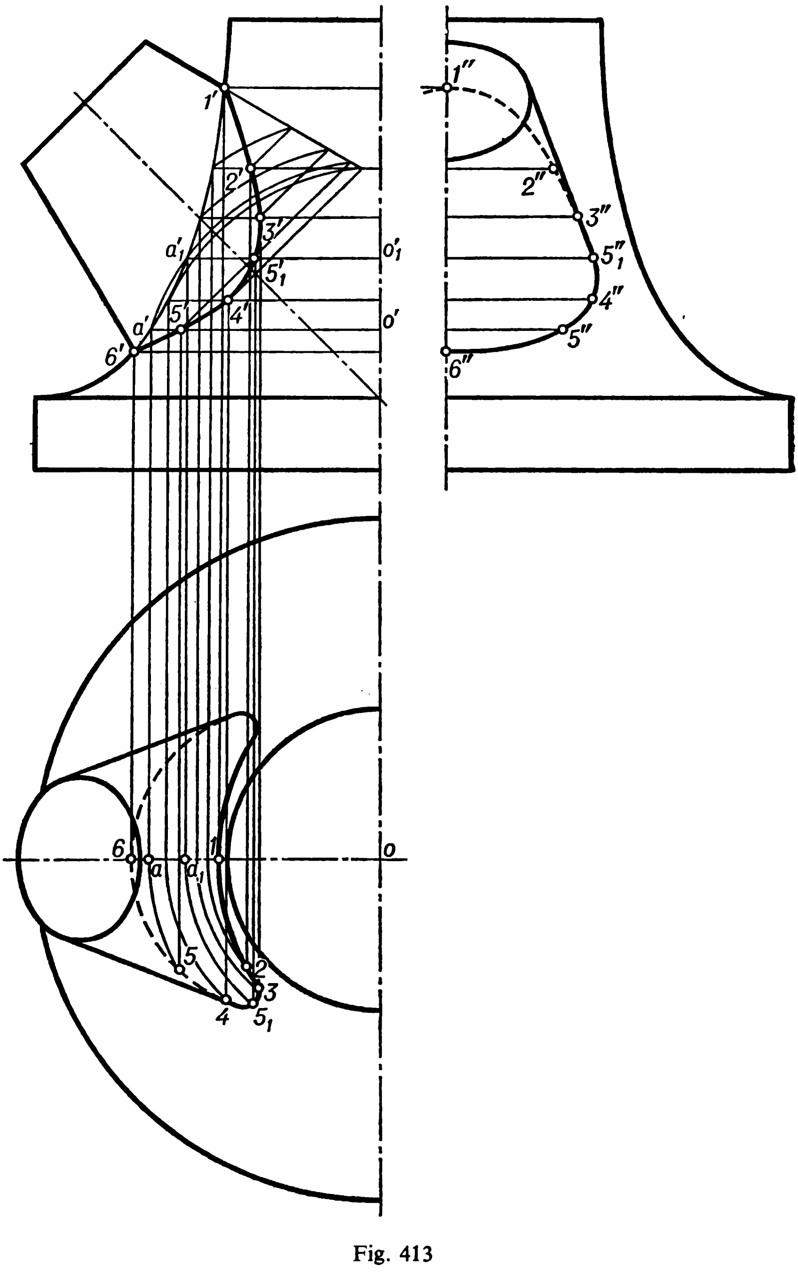


Fig. 413

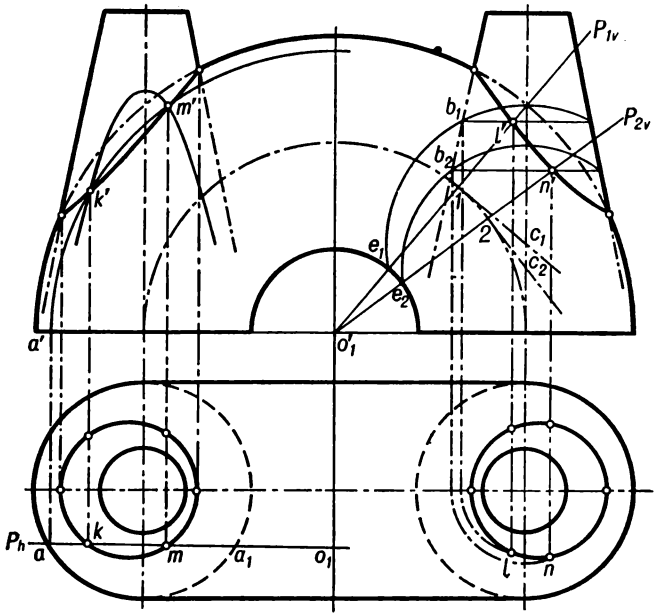


Fig. 414

through the axis of the cone. *This is an example illustrating how in one and the same construction two different methods are used, namely, the method of auxiliary cutting planes and the method of auxiliary cutting spheres.*

Figure 413 demonstrates the coupling of the surfaces of two solids of revolution: a conical and one with a curvilinear generatrix. Auxiliary spheres are used here. We determine the projections of the required points first on the *V* plane and then on the *H* plane. For instance, point *5* on the horizontal plane is determined on a circular arc of radius $oa = o'a'$ described from point o; point 5_1 is obtained on the arc of radius $oa_1 = o'_1a'_1$. The point with the projections *4'* and *4* is found with the aid of the sphere inscribed in the surface of revolution with a curvilinear generatrix.

The points on the profile plane are found in a usual way as the third projection by two determined on the vertical and horizontal planes of projection. To save space, all the views of Fig. 413 are given incompleted.

The example given in Fig. 414 makes it possible to establish the advantage of the method of auxiliary spheres over other methods as applied to the given case. It is required to construct the projections of the line of coupling the surfaces of a cone of revolution and an annular torus (Fig. 414 represents half the torus). The left-hand part of the drawing shows the use of auxiliary planes parallel to the axis of the cone. These planes cut the conical surface in hyperbolas which we have to plot point by point, and the torus in semi-circles of radii o_1a and o_1a_1. For instance, on constructing on the vertical projection a hyperbola (the line of intersection of the conical surface by a plane P), we strike a circular arc of radius $o_1'a' = o_1a$, find the points k' and m' on the vertical projection and then the horizontal projection k and m.

We have to construct a number of hyperbolas which complicates the solution and reduces the accuracy. The planes perpendicular to the cone axis are also inconvenient, since they will cut the torus surface in curves. To construct each of these curves, we would have to find a number of points (see Sec. 58). The planes passing through the vertex of the cone are also no good, since they cut the surface of the torus in curves to be plotted point by point.

The construction becomes simplified and more accurate with the use of auxiliary spheres whose centres lie on the axis of the cone. The spheres should be chosen so as to cut the torus in circles. It is done in the following way.

Let us take a plane P_1 passing through the axis of the torus and perpendicular to the vertical plane. It will cut the torus in a circle of radius $1e_1$ with centre at point 1. This circle is projected on the vertical plane into a line segment. Where must the centres of the spheres which may be drawn through this circle be located? Obviously, they lie on a straight line passing through the centre of circle 1 and perpendicular to the plane P_1. On the vertical projection this straight line is represented by a line $1c_1$ perpendicular to P_1 (and, hence, tangent to the axial circle of the torus shown in the drawing with a long-chain line).

Thus, we have to construct a sphere whose centre must lie, firstly, on the axis of the cone, and, secondly, on the straight line $1c_1$. Such a centre c_1 is quite determined by these two lines, and we may construct a sphere of radius c_1e_1 and c_1 as centre; shown on the vertical plane is a portion of the projection of the sphere, i.e. an arc of a circle. The intersection of the sphere with the cone is a circle which is projected into a line segment passing through the point b_1; and the torus is cut in the above indicated circle projected into a line segment on the trace P_{1v}. The intersection of these straight lines yields a point l' which is the projection of one of the points belonging to the required line.

Analogously, we find the point n' with the aid of the plane P_2 and the points 2, c_2, b_2, e_2. To construct the horizontal projections of these points, we may use the parallels of the conical surface, as is shown for the points l and n.

We may suppose that the lines c_11 and c_22 are the axes of some cylinders whose normal section coincides with the normal section of the torus. If we take points 1 and 2 situated quite close to each other and imagine that such points are numerous, and, hence, there are many axes drawn through these points, and thus there are many cylinders, then the surface of the torus will turn out to be replaced by cylindrical surfaces arranged in succession. Therefore the problem is reduced to finding the points common to the

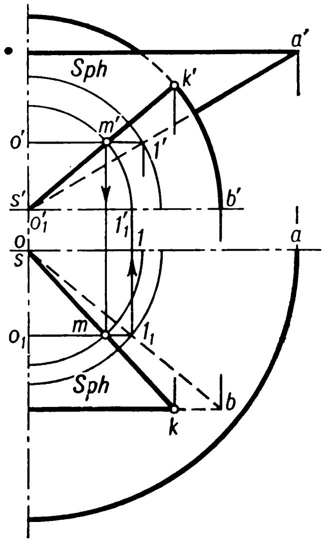


Fig. 415

conical surface and to the surface of each "instantaneous cylinder"*. The axes of "instantaneous cylinders" intersect the axis of the cone at points which are taken for the centres of auxiliary spheres cutting the cone and "instantaneous cylinder" in circles. The projections of these circles on the vertical plane represent segments of straight lines. The circles in which the auxiliary spheres cut the "instantaneous cylinders" are the normal sections of the torus with which we began the construction.

Figure 415 represents (partially) two cones of revolution with a common vertex S and shows the construction of the generating element along which the conical surfaces intersect. One point of the required element is known: this is the vertex S. The second point is found by making use of an auxiliary sphere with the point S as centre. This sphere intersects the conical surfaces along circular arcs whose radii are equal to $o1$ or $o'1'$ and $o_1 1_1$ or $o_1'1_1'$, respectively. The vertical projection of these arcs intersect at point m', and horizontal ones at point m. The points m' and m are the projections of the point M—the second point for the required generating element.

Such a construction was used in Fig. 401.

Sec. 65. Projecting the Line of Intersection of Two Second-order Surfaces of Revolution on a Plane Parallel to Their Common Plane of Symmetry

In a number of cases we come across intersections of two algebraic second-order surfaces of revolution. The line of intersection obtained is a space curve of the fourth order called *biquadratic*.

It was stated in the footnote on page 289 that the line of intersection of two second-order surfaces having a common plane of symmetry is projected on a plane parallel to the plane of symmetry into a second-order curve. Figure 412 to which this footnote referred represented two cones of revolution with intersecting axes which defined a common (for these cones) plane of symmetry parallel to the vertical plane. The vertical projection of the biquadratic curve thus obtained represented a hyperbola.

*We resort to this expression in order to stress the substitution of the surface of the torus by numerous cylindrical elements. In practice, only a few of such constructions are accomplished.

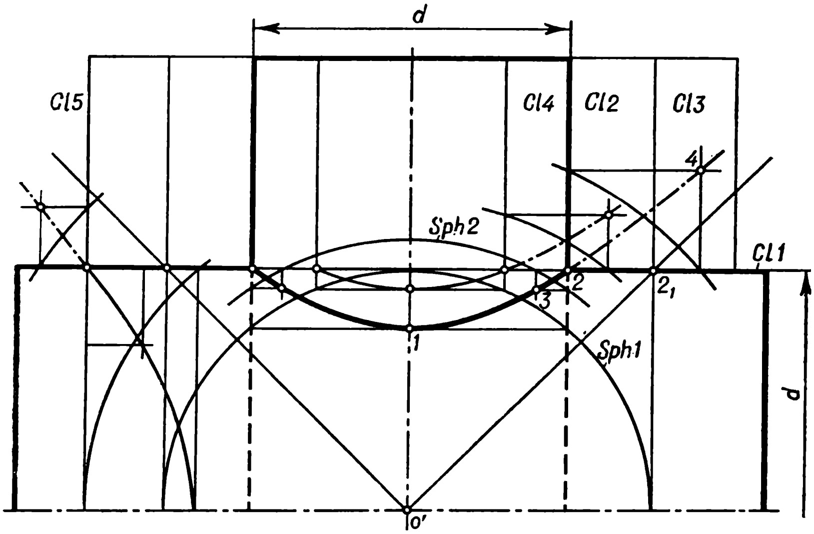


Fig. 416

Figure 416 gives the vertical projection of two cylinders of revolution of different diameters (*Cl 1* and *Cl 2*). Point o' is the vertical projection of the point of intersection of the axes of the cylinders. The vertical projection of the biquadratic curve obtained represents an equilateral hyperbola (one of its branches) with centre at the point o'. The construction is carried out with the aid of spheres having a common centre at the point of intersection of the axes of the cylinders. The sphere inscribed in the cylinder of greater diameter *(Sph 1)* enables us to find point *1* representing the vertex of the hyperbola. Spheres with greater radii yield other points of the required projection of the curve (for instance, the sphere *Sph 2* gives point *3*). If the radius is greater than the line segment $o'2$, then points are obtained (for instance, *4*) outside the area occupied by the projection of both cylinders.

Figure 416 shows the asymptotes of the constructed hyperbola; they pass through the point o' and are mutually perpendicular. These asymptotes preserve their sense for all the hyperbolas obtained in Fig. 416 if we take, for instance, cylinders of different diameters with a vertical axis (*Cl 4*, *Cl 5)*. But if the cylinders are of equal diameters (say, *Cl 1* and *Cl 3*), i.e. these cylinders have a common inscribed sphere *(Sph 1)*, then the vertical projection of the line of intersection in Fig. 416 (see also Fig. 404) represents two straight lines intersecting at right angles, whose positions (for instance, $o'2_1$) correspond to the positions of the asymptotes.

If the axes of the cylinders intersect at an acute angle (as in Fig. 417), then the projection of the line of intersection under the same conditions as in the case considered previously represents also an equilateral hyperbola. The points for this sphere are constructed using the method of auxiliary

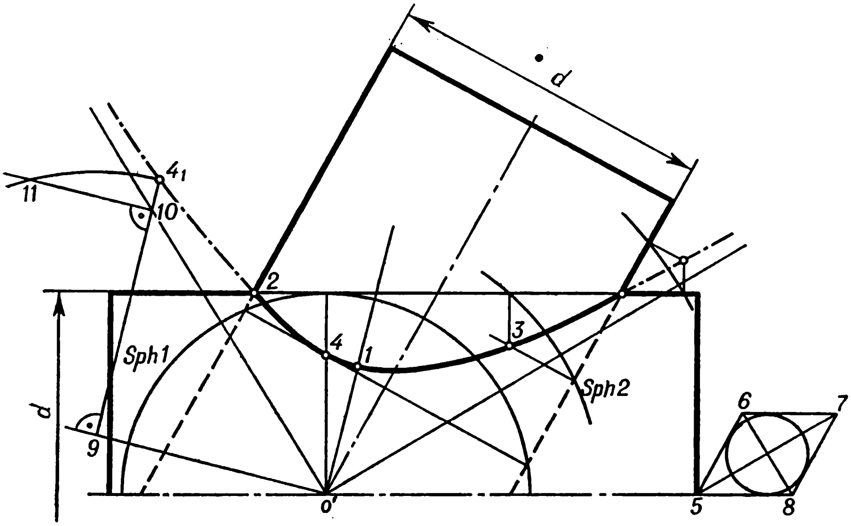


Fig. 417

spheres, and in this respect there is no difference between the cases depicted in Figs. 417 and 416. We draw the reader's attention only to the fact that point *4* obtained with the aid of the sphere inscribed in the greater cylinder *(Sph 1)* is no longer the vertex of the hyperbola as it was in Fig. 416.

Let us now follow the peculiarities of the construction shown in Fig. 417. To determine the positions of the asymptotes, a rhombus *5-6-7-8* is constructed whose sides are tangent to a certain circle and parallel to the generatrices of the cylinders. The diagonals of this rhombus indicate the directions of the asymptotes. Hence the asymptotes are mutually perpendicular which means that the hyperbola is equilateral.

Drawing the bisector of the angle between the asymptotes, we get the real axis of the hyperbola whose vertex must lie on this axis (point *1*). To find this point, we carry out the following construction: taking an arbitrary point of the hyperbola, say, point 4_1, we draw through it a perpendicular to the conjugate axis of the hyperbola and mark points *9* and *10* at which this perpendicular intersects the conjugate axis and asymptote. Then we describe a circular arc of radius $9\text{-}4_1$ to intersect (at point *11*) the perpendicular drawn from point *10* to the line $9\text{-}4_1$. The obtained line segment *10-11* represents the distance from *o'* to *1*, i.e. to the vertex of the hyperbola, or in other words, its semitransverse axis.

The line of intersection of the surfaces of revolution represented in Fig. 418 is projected on the *V* plane parallel to the common plane of symmetry of these surfaces into a hyperbola (its asymptotes are parallel to the diagonals *1-3* and *2-4* of the trapezoid whose sides are respectively parallel to the generatrices of the given surfaces and touch a circle). Moreover, in this case there

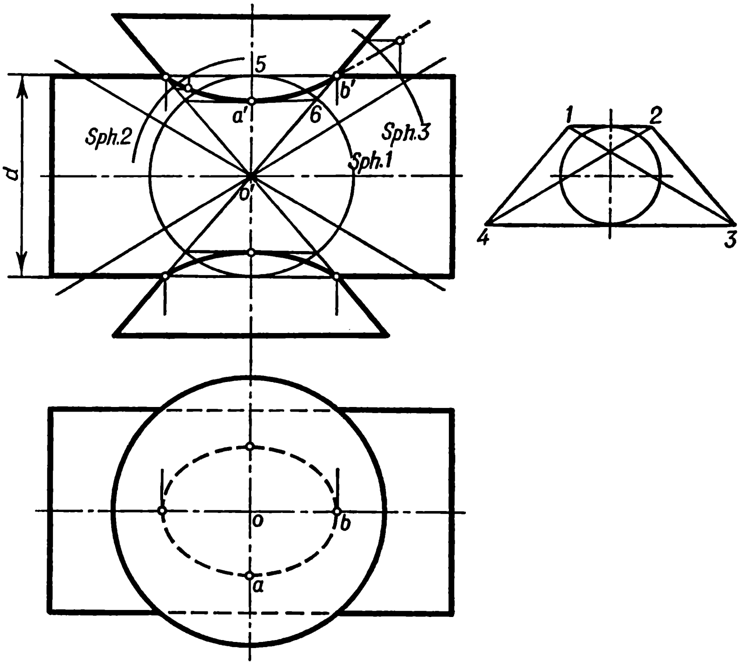


Fig. 418

is a plane of symmetry perpendicular to the axis of the conical surface. It is horizontal and passes through the axis of the cylinder. The projection of the line of intersection of the surfaces under consideration on this plane must be a second-order curve. We obtain a closed curve with two mutually perpendicular axes of symmetry, that is an *ellipse*. Its semimajor axis *ob* is equal to the line segment *b'5*, the semiminor axis *(oa)* being equal to the line segment *a'6*, i.e. to the radius of the parallel of *Sph. 1* on which point *A* is located.

The hyperbola obtained in Fig. 418 is not an equilateral one, since its asymptotes form angles not equal to 90°. The same in Fig. 419 where also a hyperbola is constructed representing the projection of the line of intersection of a cylinder with a conical surface; the hyperbola is not equilateral. This is characteristic for the cases of mutual intersection of a conical and a cylindrical surfaces of the second order having a common plane of symmetry when the line of intersection is projected on a plane parallel to the plane of symmetry.

In Figure 419 the centre of auxiliary spheres is located at point ***0*** whose vertical projection is found at the point of intersection of the axes of a conical and cylindrical surfaces. The sphere inscribed in the conical surface *(Sph 1)* enables us to obtain the position of the transverse axis, centre and vertices

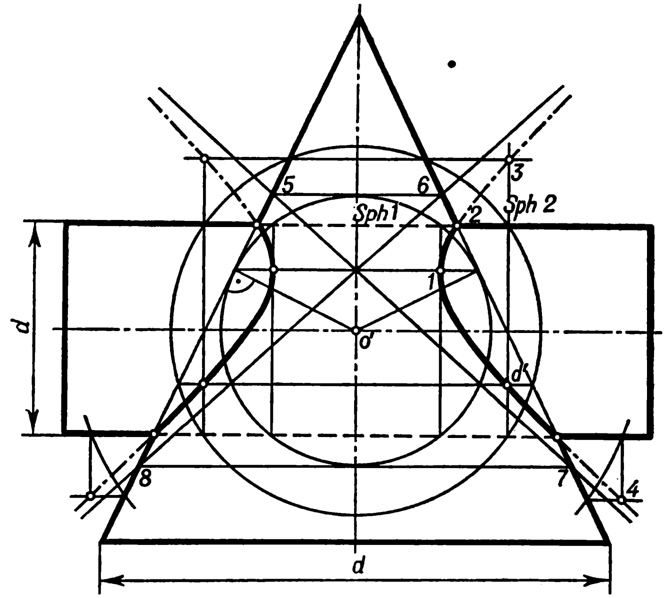


Fig. 419

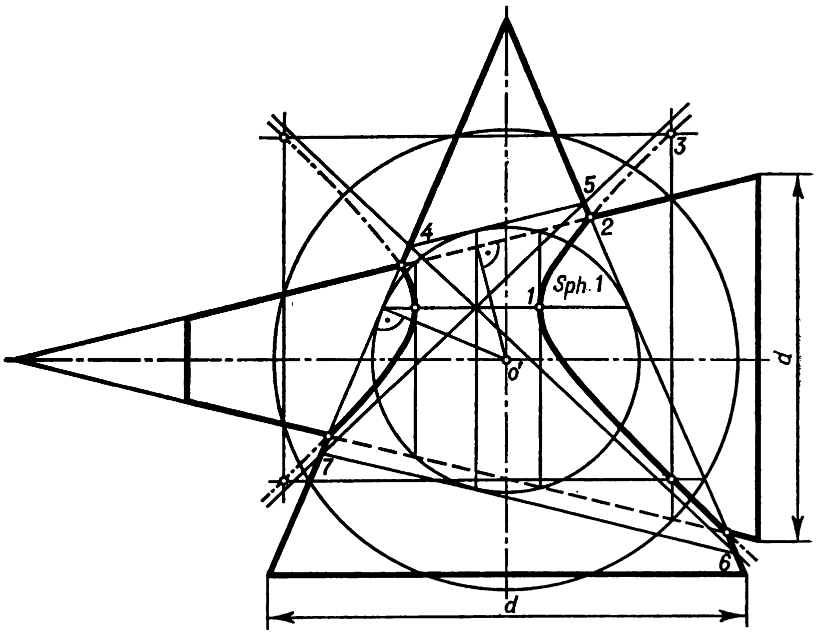


Fig. 420

of the hyperbola. The asymptotes are determined as the diagonals of the trapezoid *5-6-7-8* in which the sides *5-6* and *7-8* are parallel to the generatrix of the cylinder and touch the circle "*Sph 1*".

Thus, in Figures 416 and 417 the projections of the line of intersection represent an equilateral hyperbola; in Figs. 418 and 419 we also obtained hyperbolas but non-equilateral. A non-equilateral hyperbola is also obtained in the case shown in Fig. 420 where the projection of the line of intersection of two conical surfaces of revolution is constructed. Here the sphere *(Sph 1)* inscribed in the cone with a greater vertex angle enables us to get the position of the transverse axis, the centre and vertices of the hyperbola. The asymptotes are constructed as the diagonals of the trapezoid *4-5-6-7*.

An analogous case was demostrated in Fig. 412 representing two projections of two cones with mutually perpendicular intersecting axes.

Do we always get the projection of the line of intersection of two conical surfaces in the form of a non-equilateral hyperbola? No, we do not. If the vertex angles of the cones represented in Figs. 412 and 420 are equal to each other, then the hyperbola obtained as the projection of the line of intersection of two conical surfaces of revolution with intersecting axes on a plane parallel to these axes will turn out to be equilateral.

Tabulated below are the findings on projecting the line of intersection of two second-order surfaces of revolution with intersecting axes on a plane parallel to these axes borrowed from Glazunov's study.

The projection obtained	Surfaces of revolution	
	without any special conditions	with additional conditions
Hyperbola	Cylindrical, Conical, Paraboloids, Hyperboloids, Prolate ellipsoids — in any combination	Both surfaces are oblate ellipsoids
Equilateral hyperbola	Both surfaces are cylindrical Both surfaces are paraboloids A cylinder and a paraboloid	Both surfaces are conical with equal vertex angles Both surfaces are hyperboloids with equal angles at the vertices of their asymptotic cones A cone and a hyperboloid with equal angles at the vertices of the cone and asymptotic cone of the hyperboloid Both surfaces are ellipsoids, but similar ones

Let us return to Fig. 411 which represented the construction of the vertical projection of the line of coupling of the surfaces of a cylinder of revolution and a sphere. The common plane of symmetry of the surfaces defined by the cylinder axis and the centre of the sphere is parallel to the *V*

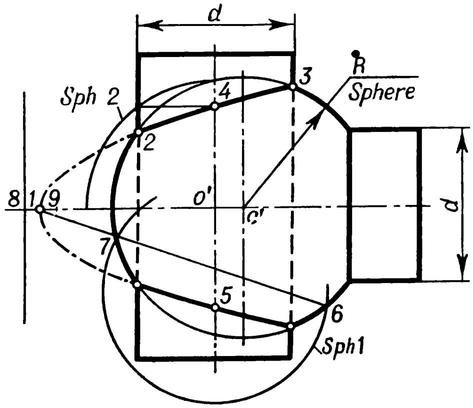


Fig. 421

plane. Therefore the vertical projection of the line of coupling of the given surface represents a second-order curve. In the case under consideration it is a parabola with the vertex at the point b'.

Figure 421 shows the construction of a parabola which represents the projection of the line of intersection of a sphere and a cylinder. Obviously, points *2* and *3* (and ones symmetrical to them) belong to the required projection. Point *4* is constructed with the aid of a circle described from the point o'. This circle is the principal meridian of the sphere *(Sph 2)* whose centre is located on the cylinder axis at the point O.

To construct point *1* (the vertex of the parabola), an auxiliary sphere *(Sph. 1)* is taken. The point is found as the intersection of the line *6-7* with the projection of the parabola axis. The Soviet scientist E.A. Glazunov established that the parameter of the parabola is equal to the distance between the points o' and c'. Laying off half the length of this segment on both sides of the parabola vertex on its axis, we get points *8* and *9*. The directrix of the parabola passes through the point *8*, its focus being located at the point *9*. Using the found directrix and focus, we may now construct the points of the parabola.

If the diameter of the cylinder intersecting the sphere is equal to its radius, and the generatrix of the cylinder passes through the centre of the sphere (Fig. 422), a biquadratic curve is obtained which is called the curve of Viviani*. Its vertical projection is a parabola.

The projection on a plane parallel to the other plane of symmetry (see the right-hand drawing of Fig. 422), i.e. on the H plane in the present case, coinciding with the projection of the cylinder, represents a circle, i.e. a second-order curve. This conforms to the general rule formulated at the beginning of this section.

*Vincenzo Viviani (1622-1703), mathematician and architect, pupil of Galileo. Viviani used this biquadratic curve for windows in a spherical dome.

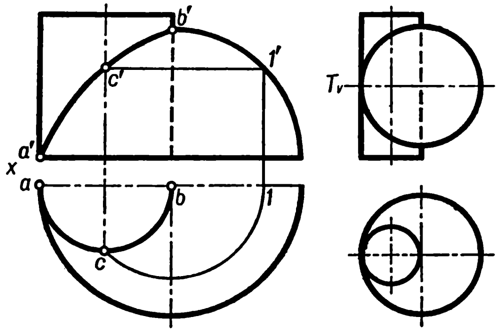


Fig. 422

Each diametral plane of the sphere is a plane of symmetry. If a second-order surface of revolution intersects a sphere whose centre is contained in the plane of symmetry of this surface, then the line of intersection is projected on a plane parallel to the plane of symmetry into a curve of the second order. We have come across this event in Figs. 418 and 422. If we constructed the horizontal projection in Fig. 421, then the curve of intersection of the cylinder with the sphere would be projected into a circle which is also obvious as in Fig. 422. Still earlier, in Fig. 398, the vertical projection of the curve of intersection of a cone with the surface of a hemi-sphere represented a parabola, and the profile projection an ellipse. We have to imagine the second hemi-sphere and the second cone in the same relative positions as in Fig. 398, and to bring both hemi-spheres together to form a sphere. The plane of contact will then appear as a pronounced plane of symmetry parallel to the W plane, and the curve on W as an ellipse. Figure 399 also illustrated the parabola and ellipse as the projections of a line of intersection.

The below table indicates in what cases the intersections of two second-order surfaces of revolution with intersecting axes are parabolas and ellipses obtained as the projections of the line of intersection on planes parallel to the plane of symmetry of these surfaces*.

The projection obtained	Surfaces of revolution
Parabola	A sphere with a cylinder, cone, paraboloid, hyperboloid, ellipsoid
Ellipse	An oblate ellipsoid with a cylinder, cone, paraboloid, hyperboloid, prolate ellipsoid

*Borrowed from the mentioned study.

Knowing for sure what line is to be obtained as a result of the construction, in a number of cases we may take advantage of the geometric properties of these lines which simplifies the constructions involved and makes it possible to get more accurate results.

QUESTIONS TO SECS. 63-65

1. What are the lines of intersection of (a) two cylindrical surfaces whose generatrices are parallel to each other, (b) two conical surfaces with a common vertex?

2. How do we construct the generatrices of a ruled surface called a cylinder with three directrices if two of them or all three are curved lines?

3. What lines are yielded by mutual intersection of two surfaces of revolution circumscribed about a common sphere or inscribed in a sphere?

4. What are the lines of intersection of coaxial surfaces of revolution?

5. In what cases is it possible and advisable to make use of auxiliary cutting spheres?

6. What curve is called biquadratic?

7. Into what line is a biquadratic curve projected on a plane parallel to a common plane of symmetry of two intersecting surfaces of the second order?

8. What second-order curve is the projection of the line of intersection of two cylindrical surfaces on a plane parallel to a common plane of symmetry of these surfaces?

9. In what case the projection of the line of intersection of two conical surfaces having a common plane of symmetry parallel to the plane of projection is an equilateral hyperbola?

10. What curve may represent the projections of the line of intersection of the surfaces of a cylinder and a cone of revolution with a sphere if they have a common plane of symmetry?

Sec. 66. Examples Illustrating the Construction of the Line of Intersection of Two Surfaces

Considered below are several examples illustrating the use of the construction methods discussed in the previous sections as also some special methods suitable for constructing the points of the required line of intersection of surfaces occupying particular positions.

In Figure 423 the projection of the line of intersection on the H plane coincides with the circle representing the projection of the cylinder with a vertical axis, and on the W plane with the semicircle which is the projection of the cylinder with a horizontal axis. Finally, we have to find the points with the aid of which it is possible to construct the projection of the required line on the V plane (hyperbola with the vertex at point b').

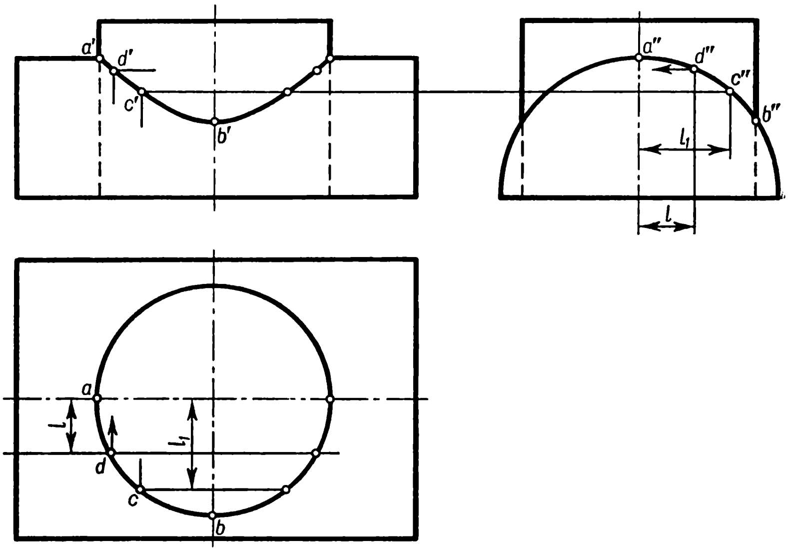


Fig. 423

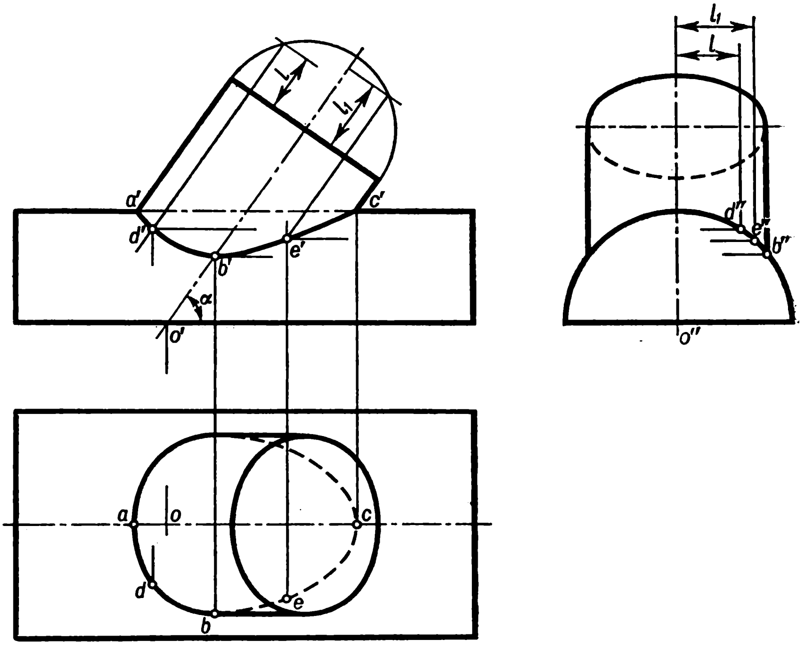


Fig. 424

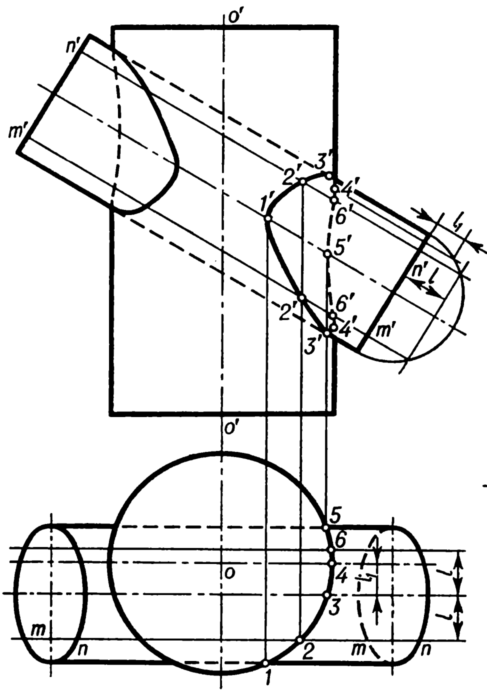


Fig. 425

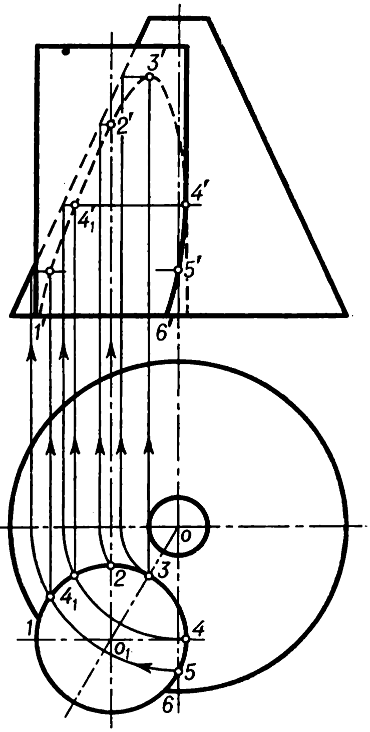


Fig. 426

Obviously, the projection b' is determined directly by the projection b'', and, for instance, the projection d' is determined as the point of intersection of the lines of recall drawn from the points d and d'' coordinated by the distance l from the axes of the horizontal and profile projections.

The projection c' is also determined by the coordinated projections c and c''. As is clear, there is no need here to pass auxiliary planes or spheres.

In Figure 424 the projections b', d', e' are constructed with the aid of the profile projections b'', d'', e'' which enable us to find the vertical projections of the elements of the oblique cylinder and the projections b', d', and e'. Once the projections b'', d'', e'', a', b', c', d', e' are determined, we can find the projections a, b, c, d, e.

In the case demonstrated in Fig. 425 the points for the vertical projections of the lines along which the oblique cylinder intersects the surface of the cylinder with a vertical axis are found proceeding from the positions of the horizontal projections of these points. We have only to construct the vertical projections of the corresponding elements of the inclined cylinder. Of points marked in Fig. 425, the following ones are regarded as reference points: *1'* and *5'* are the nearest points to the axis of the vertical cylinder on the

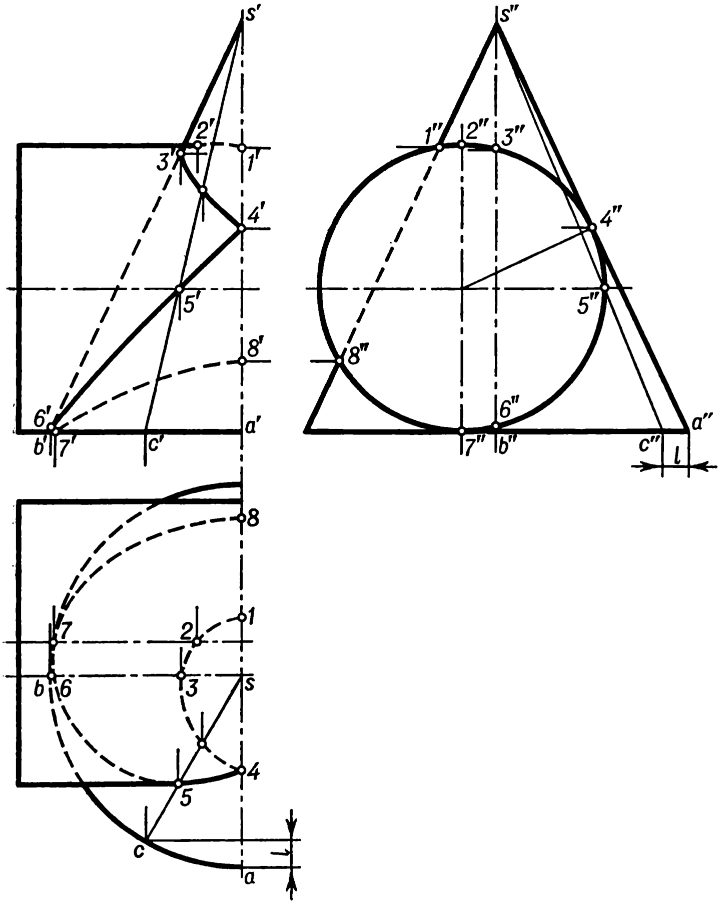


Fig. 427

visible and invisible portions of the vertical projection of the right-hand line of intersection, respectively; *3′* and *3′* are the uppermost and lowest points (respectively) on the extreme elements of the inclined cylinder; *4′* and *4′* are the points separating the projection of the extreme element of the vertical cylinder from the projection of the curve. These points correspond to their counterparts on the left-hand curve.

Figure 426 shows a cylinder intersecting the surface of a cone. In this case the points *1′*, *2′*, . . ., *6′* are constructed from the points *1*, *2*, . . ., *6* of the horizontal projection of the line on the conical surface. For instance, points *4′* and $4'_1$ are obtained on the vertical projection of the parallel of radius *o4*, and the point *3′* on the vertical projection of the parallel of radius *o3*.

The vertical projection of the line of intersection of a cylindrical surface and a cone (see Fig. 427) is constructed with the aid of initial points taken on the profile projection of the cylinder. Points *1''*, *3''*, *4''*, *6''*, *8''* enable us to immediately find the reference points *1'*, *3'*, *4'*, *6'*, and *8'* for the vertical projection. Other points can be found with the aid of generating elements. For instance, taking the projection *s'c'* of the element on which the projection *5''* must lie, we find, using the line segment *l*, the point *c* and the projection *s''c''*, and then *s'c'*. What is left is to get the projections *5'* and *5*.

In Figure 428 the vertical projections of the points of the line along which the cylinder cuts the surface of a hemi-sphere can be found from the horizontal projections on the corresponding parallels of the sphere. For instance, using the point *k*, we determine the parallel of radius *ok* and on its vertical projection find the projection *k'*. The same is shown for the points *A* and *F*. But, of course, for the points *A* and *F*, proceeding from the positions of their horizontal projections *a* and *f*, we may take a cutting plane *T* parallel to *V*, and find the projections *a'* and *f'* on the semicircle in which the plane *T* cuts the surface of the hemi-sphere. Obviously, in most cases it is expedient to vary the methods for constructing the points belonging to the projections of the lines of intersection, choosing the most convenient methods in order to get simple and accurate constructions.

In Figure 428 the projections *b'* and *e'* are found on the principal meridian directly using the points *b* and *e*. We could also find the projections *d'* and *g'* if we had the profile projection. Now, with the profile projection absent, we can find the points *d'* and *g'* in the same manner as, for instance, the projections *a'* and *f'*.

The projections *a*, *b*, *c*, etc. marked in Fig. 428 determine the reference points for the vertical projection of the curve and for the profile projection if it is to be constructed: *k'* and *m'* are the lowest and uppermost points respectively; at points *b'* and *e'* the principal meridian is "discontinued" on the sphere and at points *a'* and *f'* the line of intersection is separated into a visible and invisible parts; the points *d'*, *g'*, *c'*, *h'*, being of no special importance for the vertical projection, enable us to construct the reference points on the profile projection of the curve.

Figure 429 represents a solid of revolution with a cylindrical hole made in it. The curve *k'a'b'm'* is constructed, using the points *k*, *a*, *b*, *m*, i.e. the known horizontal projections. For instance, taking the point *a*, we construct the projection of a parallel on the surface of revolution, and find the projection *a'* on the vertical projection of this parallel.

To construct the vertical projection of the line of coupling of an annular torus and a cylinder (Fig. 430) we use the horizontal projections of the points (in the same way as in Fig. 429). For instance, knowing the position of the point *b*, we can draw on the surface of the torus circular arcs of radii $o2$ and $o2_1$ and obtain points *b'* and b_1' on these arcs. Here a system of circular sections of the torus surface is used.

In Figure 431 we also take advantage of the fact that the positions of the points of one of the projections of the required line are known. This enables

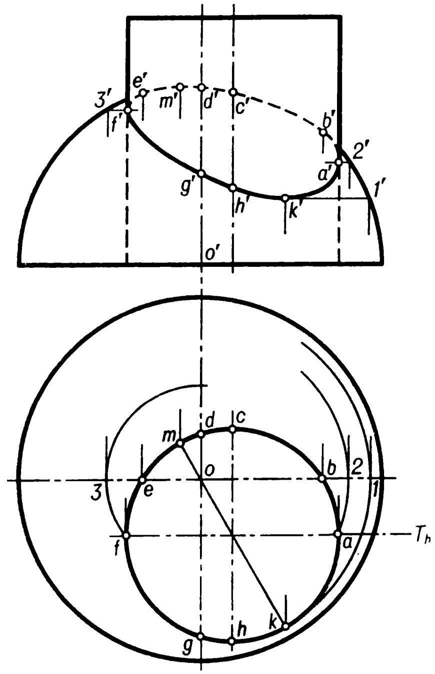


Fig. 428

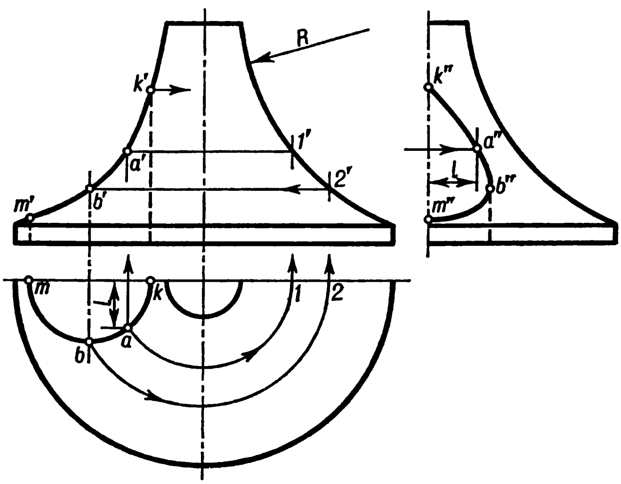


Fig. 429

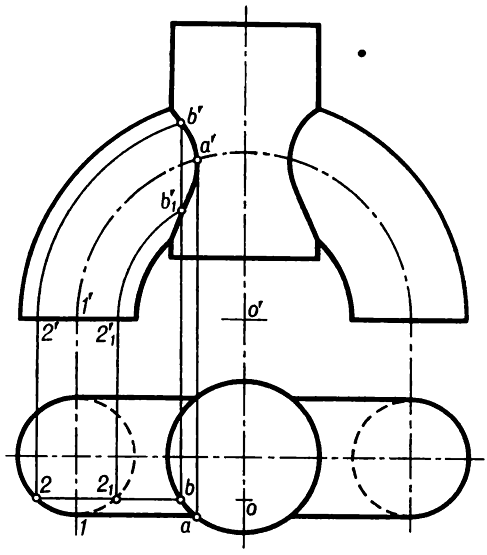


Fig. 430

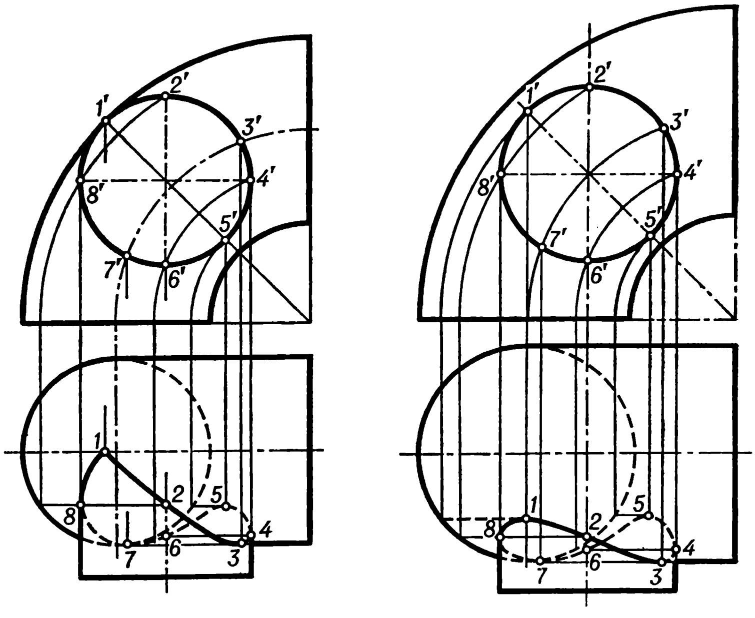
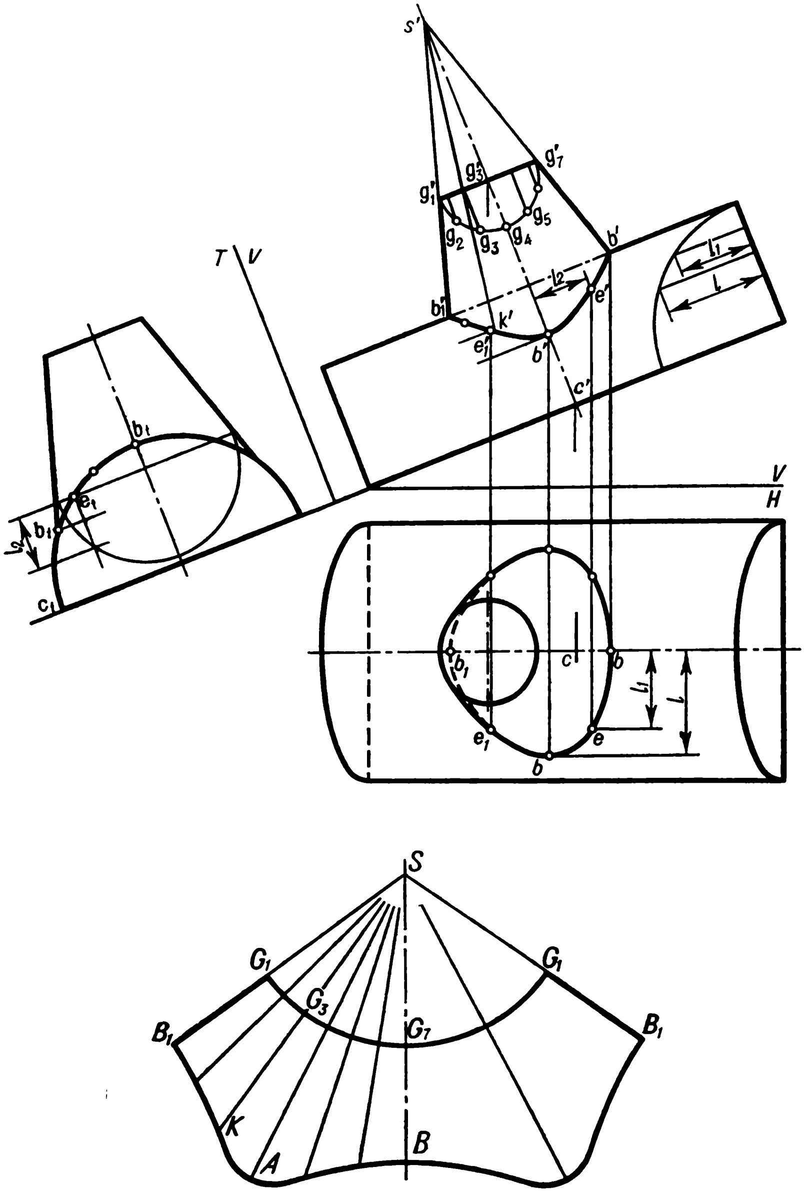


Fig. 431



s'
g_1'
g_3'
g_7'
g_2
g_3
g_4
g_5
b'
T
V
l_1
l
l_2
e'
b_1''
k'
e_1'
b'
c'
b_t
e_t
b_t
c_t
V
H
b_1
c
b
l_1
l
e_1
e
b
S
G_1
G_3
G_7
G_1
B_1
B_1
K
A
B

Fig. 432

us to construct the points belonging to the other projections. In the left-hand drawing of Fig. 431 a corner point (point of break) is obtained on the horizontal projection.

The construction of the vertical projection of the curved line of intersection of a conical and cylindrical surfaces represented in Fig. 432 could be carried out, for instance, in the manner shown in Fig. 419, i.e. with the aid of spheres with the point C as centre. On constructing the hyperbola, we can construct the horizontal projection of the curve with the aid of elements of the cylinder; for instance, the element on which point E is located is determined by the line segment l_1.

Figure 432 shows another method of construction which consists in *projecting on an additional plane*, in the present case on a vertical projecting plane perpendicular to the axis of the cylindrical surface. The line of intersection is projected on this plane in the form of an arc on the semicircle representing the projection of this surface. Taking points on the arc, we can construct their horizontal and vertical projections. For example, taking the point e_t, we determine the line segment l_2 on the semicircle of radius R representing half of the parallel on the cone. Laying off the line segment l_2 (as is shown in the drawing) on the vertical projection, we get the projection e' on the line of recall with the projection e_t.

Figure 432 also shows the development of the lateral surface of the frustum of a cone considered in this problem. The procedure of its construction is as follows: We construct the projection of the cone vertex (point s'); revolve the circumference of the upper base of the cone to occupy the position in which it is parallel to the vertical plane, and divide it into several parts (the drawing gives half of this circle). Projecting the points g_2, g_3, etc. on the straight line $g_1'g_7'$, we draw through these projections and through s' the projections of the elements to meet the projection of the line of intersection of the surfaces; for instance, $s'k'$ is drawn through g_3'.

On constructing the development of the lateral surface of the cone, we lay off the lengths of the segments of the corresponding elements. For instance, finding (by the method of revolution) the length of the segment of the element G_3K, we lay it off on the corresponding element on the development.

Constructed in Fig. 433 is the line of intersection of a quadrangular prism with a cylinder and the development of the obtained portion of the prism.

Each face of the prism cuts the cylindrical surface in an ellipse. The ellipses thus obtained intersect one another at points which are the points of intersection of the faces of the prism with the cylindrical surface. The vertical projections of the mentioned points are determined by their profile projections. For any point E we determine the horizontal projection e by its profile projection e''. Knowing the projections e'' and e, we find e'. Points a' and b' are determined by the horizontal projections of the points A and B.

To construct the development of the prism, we divide the horizontal

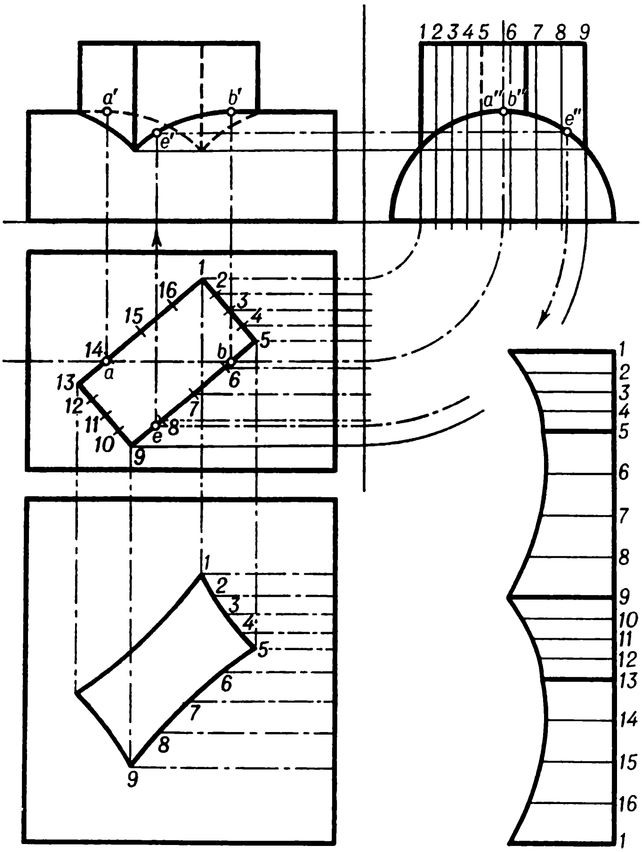


Fig. 433

projection of the prism into line segments, taking an equal number of divisions on each face. This division corresponds to the division of the cylindrical surface within the area of its intersection by the prism.

Figure 434 represents the construction of the line of intersection of a pyramid with a cylinder and the developments of both surfaces.

The lines of intersection are ellipses which intersect one another at the points where the edges of the pyramid cut the surface of the cylinder. The point b' may be constructed in the way shown in the drawing, i.e. without the aid of the profile projection.

To construct the developed surfaces of the pyramid and cylinder, the circle on the horizontal projection of the cylinder is divided into twelve equal parts. The points belonging to the ellipses are found by drawing auxiliary straight lines through the vertex of the pyramid on the development of

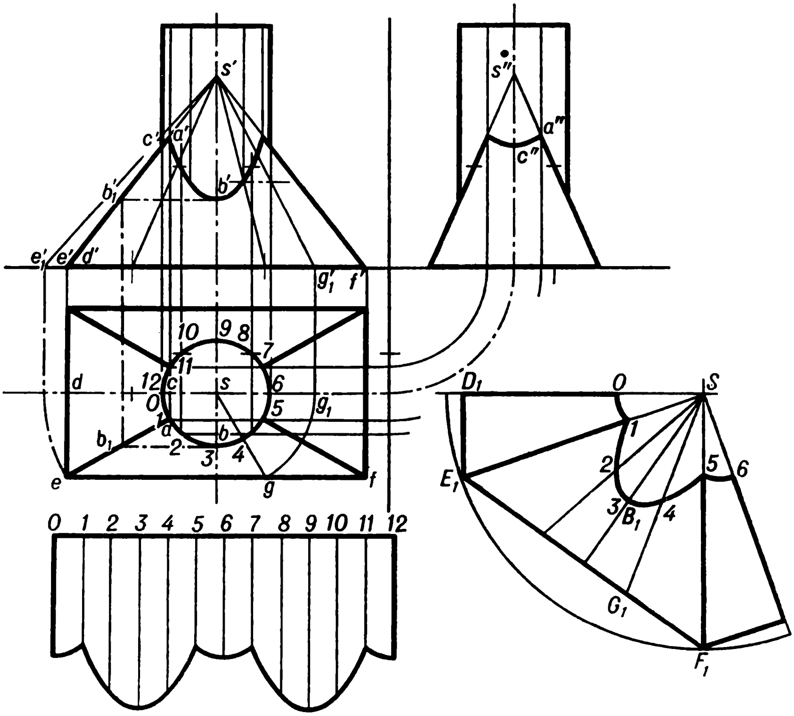


Fig. 434

the pyramid surface (for instance, the line SG). The lengths of the segments of these lines (say, $E1$) are determined by revolving them to the position parallel to the V plane.

An example of construction of the line of intersection of a prism with a sphere and the development of the surface of the prism is given in Fig. 435. The faces of the prism cut the spherical surface in circular arcs. The projections of these arcs on the H plane are parts of ellipses. The projection of the line of intersection on the V plane consists of parts of ellipses, circular arcs (since two faces of the prism are parallel to the V plane), and a straight line. We find the points of intersection of the edges of the prism with the sphere. Then we have to mark the points belonging both to the line of intersection of the prism with the sphere and to the principal meridian of the sphere. The plane determining the principal meridian cuts the prism along a straight line on which the mentioned points must be located. The drawing shows the development of the prism. The curve on the development is made up of circular arcs. Some of the radii for striking these arcs are taken from the vertical projection (R_2, R_3, R_4), the other are found with the aid of an additional projection (R_1 and R_5).

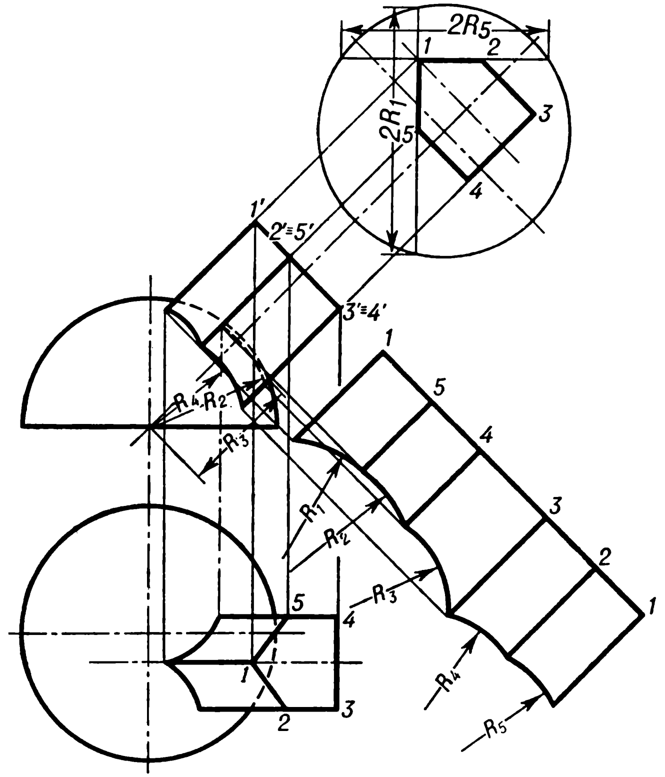


Fig. 435

Sec. 67. The Intersection of a Curved Line with a Curved Surface

To find the points of intersection of a curved line with a curved surface, it is necessary to pass an auxiliary surface through the curved line, to construct the line of intersection of the auxiliary and the given surfaces, and to find the points of intersection of this line with the given curved line.

Let us consider several examples of intersection of a space curve (a curve of double curvature) with a curved surface.

1. Figure 436 shows the construction of the point of intersection of a curve *AB* with a cylindrical surface specified by the horizontal trace *MN* and the direction of the generatrix *NP*.

Through the curve *AB* we pass an auxiliary cylindrical surface whose elements are parallel to *NP*. With the elements so directed, the line of intersection of both surfaces will be an element common to them. We then construct the trace of the auxiliary cylindrical surface on the *H* plane (the curve A_0B_0). The intersection of the curves *MN* and A_0B_0 yields a

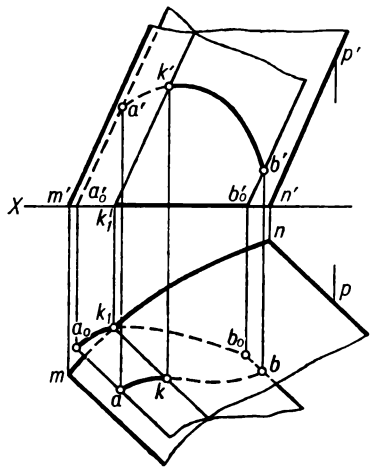


Fig. 436

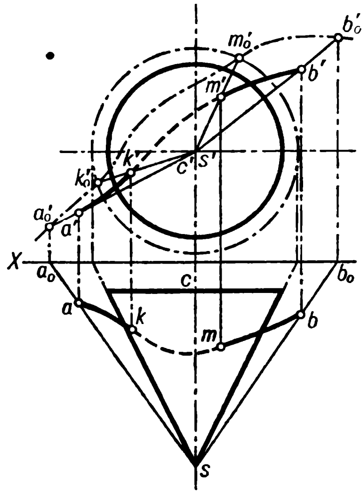


Fig. 437

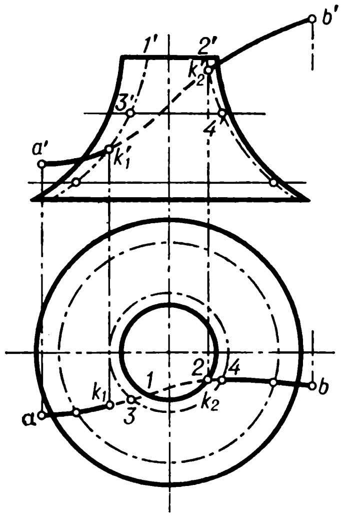


Fig. 438

point K_1 through which the line of intersection of the surfaces (their common element) passes. This element intersects the given curve AB at point K which is the required point of intersection of the line AB with the given cylindrical surface.

2. Figure 437 illustrates the construction of the points of intersection of a curve AB with a conical surface. Through the curve AB we pass an auxiliary conical surface whose vertex coincides with the vertex S of the given cone. The intersection of the two conical surfaces so arranged yield straight lines serving as common elements to both surfaces (see Sec. 63).

We construct the traces of the given and auxiliary conical surfaces. The intersection of these traces yield points K_0 and M_0 which determine the elements SK_0 and SM_0. These elements intersect the curve AB at points K and M which are the required points of intersection of this curve with the given conical surface.

3. Figure 438 demonstrates the construction of the points of intersection of the curve AB with the surface of an annular torus. Through the curve AB we pass an auxiliary cylindrical surface whose elements are perpendicular to the H plane. We then find the line of intersection of this surface with the given surface for which purpose we pass a number of planes cutting the given surface in parallels. Since the elements of the auxiliary cylindrical surface are perpendicular to the H plane, the intersections of the horizontal projections of the parallels and ab yield points ($1, 2, 3, \ldots$) which are the horizontal projections of the points determining the line of intersection of the given and auxiliary surfaces. Constructing the vertical projection of this line, we get the projections k_1', k_2', and then the projections k_1, k_2.

QUESTIONS TO SECS. 66-67

1. Indicate the methods used for the construction of the projections of the line of intersection of two surfaces.

2. How can we use the case when one of the projections of the line of intersection coincides with the projection of a cylindrical surface?

3. How should one proceed if it is required to find the point (points) of intersection of a curved line with a curved surface? In particular, if a curve intersects a cylindrical or conical surface?

CHAPTER 11

THE DEVELOPMENT OF CURVED SURFACES

Sec. 68. The Development of Cylindrical and Conical Surfaces

The development of the lateral surface of a right circular cylinder was shown in Fig. 305. The base of the obtained rectangle is then equal to the developed circumference (πd), and its altitude to the altitude of the cylinder. Figure 362 represents the development of the surface of a right circular cylinder cut by a plane in an ellipse. Its lower base is a circle which is a normal section of the cylindrical surface of revolution. The circle is developed into a straight line which is divided into a certain number of equal parts corresponding to the division of the circle in Fig. 361. Then the scheme for developing the surface of a prism is applied. The cylindrical surface is replaced here by the surface of a prism inscribed in it. The edges of the prism are equal to the segments of the elements of the cylindrical surface. Thus, we resort here to the method of approximation widely used in various branches of mathematics.

Theoretically, the more faces the prism inscribed in the cylinder has, the more exact is the development of the cylindrical surface. And the smaller is each of the segments of the polygonal line bounding the development of a prismatic surface, the more exact is the corresponding approximation.

The conical surface is developed in the general case according to the scheme for developing the surface of a pyramid. In Figure 308 the lateral surface of a right circular cone was developed using the construction known from solid geometry. The angle of the sector representing the required development is computed according to the formula $\varphi = \frac{R}{L} \cdot 360°$, where R is the radius of the base circle of the cone, and L is the length of its generatrix.

Let us consider the construction of the development of the lateral surface of an oblique cone with a circular base (see Fig. 439).

The circle of the base is substituted here by a polygon whose sides are A_1A_2, A_2A_3, and so on, and the conical surface by the surface of a pyramid

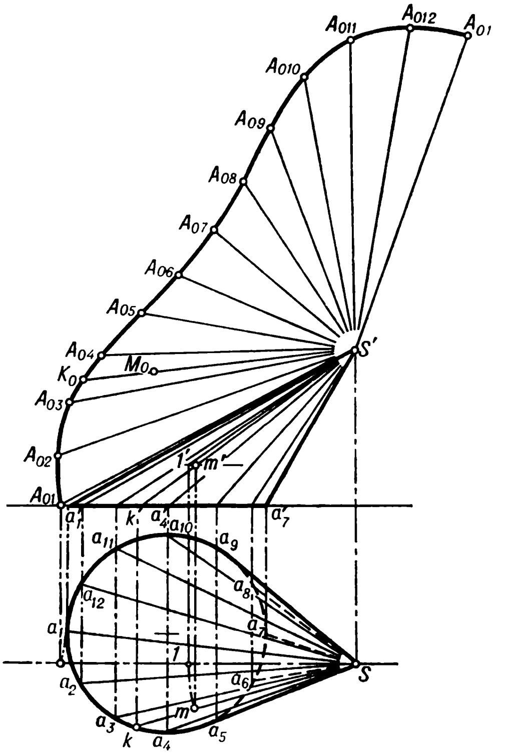


Fig. 439

with triangular faces SA_1A_2, SA_2A_3, etc. The development of this surface represents the sum of these triangles.

On having determined (by the method of revolution) the length of the line segment SA_1 (segment $s'A_{01}$) and the length of the segment SA_2 (segment $s'A_{02}$), we construct a triangle by its three sides $s'A_{01}$, $s'A_{02}$ and a_1a_2 (the chord), then a second triangle $s'A_{02}A_{03}$ for which purpose we determine the length of the segment SA_3 (segment $s'A_{03}$) and take the chord a_2a_3, and so on. Proceeding in such a way we obtain the points A_{01}, A_{02}, etc. through which we draw a smooth curve.

If it is required to find on the development a point specified on the surface, for instance M (m', m), proceed as follows: draw through this point an element $s'k'$, sk, find its position on the development ($s'K_0$), and lay off on $s'K_0$ a segment $s'M_0$. To construct the segment $s'K_0$ on the development we have to intersect the curve $A_{01}A_{02}A_{03}$... with an arc of a circle of radius a_3k described from A_{03} as centre, and to draw a straight line through the point K_0 thus obtained and point s'. Furthermore, the line segment $s'M_0$ represents the true length of the segment $s'm'$, sm after the latter is revolved to the new position $s'1$, $s1$. We finally get $s'M_0 = s'1'$.

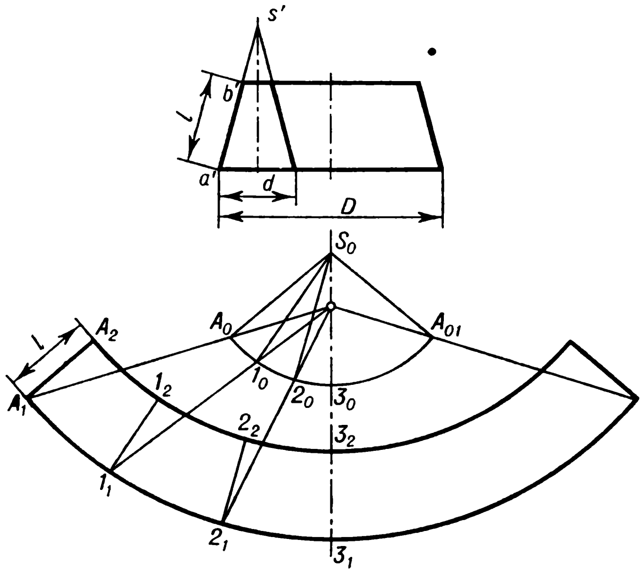


Fig. 440

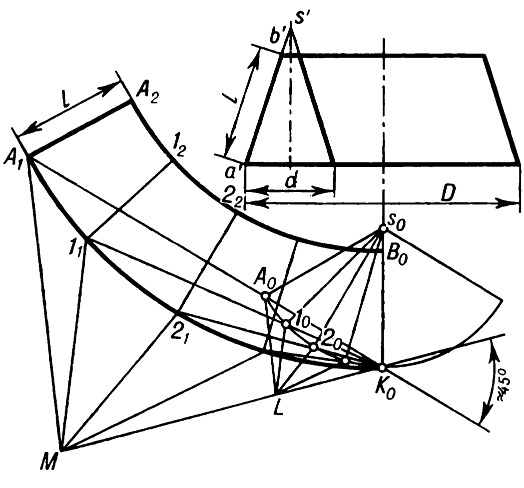


Fig. 441

We may also formulate an inverse problem: Construct the projections of a point M specified on the development (M_0). In this case we begin with drawing through the point M_0 the line segment $s'K_0$. Then we find point k on the base circle of the cone bearing in mind that $A_{03}K_0$ is equal to a_3k. On having constructed the projections sk and $s'k'$ of the generating element, we find the projections of the line segment SM for which purpose we revolve SK to a position enabling us to project it without distortion (for instance, parallel to the V plane), lay off in this position the length $s'M_0$ of the segment ($s'1' = s'M_0$), and return it to the initial position.

Figure 440 shows the development of the lateral surface of a frustum of a cone when the frustum cannot be completed to a full cone.

In this case we construct an auxiliary cone similar to the given one. It is advisable to choose the diameter d of the base of the cone so that the ratio $\frac{D}{d}$ is expressed by a whole number (k). An auxiliary cone may be constructed as is shown in Fig. 440, or outside the frustum.

Now we construct the development of the lateral surface of the auxiliary cone—the sector $S_0A_0A_{01}$: an arbitrary point K is chosen, rays KA_0, $K1_0$, $K2_0$, $K3_0$ are drawn through the points of division on the arc A_0A_{01}, and on them the following segments are laid off: $KA_1 = k \cdot KA_0$, $K1_1 = k \cdot K1_0$, $K2_1 = k \cdot K2_0$, $K3_1 = k \cdot K3_0$, where $k = \frac{D}{d}$. Then, through the points A_1, 1_1, 2_1 draw straight lines respectively parallel to S_0A_0, S_01_0, S_02_0 and lay off on them the segments $A_1A_2 = l$, $1_11_2 = l$, $2_12_2 = l$. The segment $3_13_2 = l$ is laid off in the same way. Finally, draw smooth curves through the points A_1, 1_1, 2_1, 3_1 and through the points A_2, 1_2, 2_2, 3_2.

The second half of the development may be constructed in the same way or taking advantage of the symmetry about the axis S_03_1.

Figure 441 demonstrates an alternate version of constructing the development suggested by K. Beschastnov. Here, as in Fig. 440, an auxiliary cone is also taken (in Fig. 441 the ratio $\frac{D}{d}$ is equal to three), and its development is constructed (half of it is shown in the drawing). Then, from the point K_0 several rays are drawn through the points A_0, 1_0, 2_0, ..., and a straight line K_0M is constructed at an angle of $\approx 45°$ to K_0A_1. On this line points L and M are taken so that $K_0M : K_0L$ is equal to three (i.e. to the taken ratio of D and d). Now segments LA_0, $L1_0$, $L2_0$, ... are constructed, and through the point M straight lines MA_1, $M1_1$, ... are drawn parallel to LA_0, $L1_0$, ..., respectively. The intersection of these lines with the rays K_0A_0, K_01_0, K_02_0, ... yields points A_1, 1_1, 2_1, ..., through which A_1A_2, 1_11_2, ... should be drawn parallel to S_0A_0, S_01_0, ..., respectively, and $A_1A_2 = l$, $1_11_2 = l$, etc. and also $K_0B_0 = l$ laid off.

To complete the development draw smooth curves through the points A_1, 1_1, 2_1, ..., and A_2, 1_2, 2_2, ... with the aid of a French curve, and construct the second half of the development which is symmetrical to the first about the line S_0K_0.

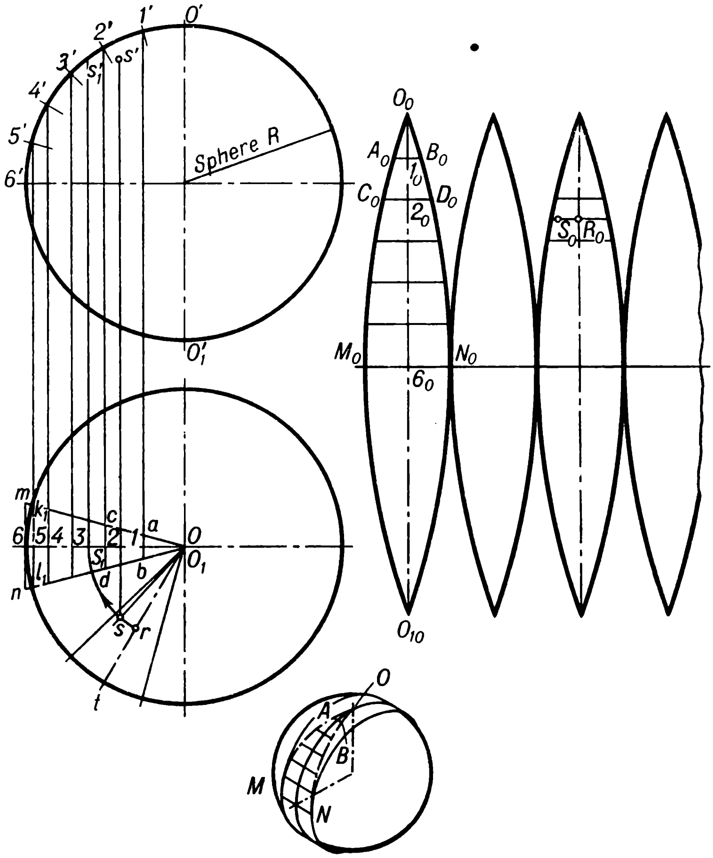


Fig. 442

Sec. 69. The Development of the Sphere

A spherical surface is undevelopable (see Sec. 49, item 5). Existing methods for constructing its development yield only approximate results. In other words, we may speak here of a conventional development.

Figure 442 shows one of the methods for constructing such a development.

1. The surface is "cut" by planes passing through the axis of the sphere OO_1 (for instance, in Fig. 442 into 12 equal parts, the vertical projections of the lines of intersection are not shown).

2. On the H plane the arcs between the points of division are replaced by straight lines tangent to the circle (for instance, the arc $k_1 6 l_1$ is replaced by the line segment mn).

3. Each portion (or element) of the spherical surface is replaced by a cylindrical surface of revolution with the axis passing through the centre of the sphere parallel to the tangent line to the great circle (the radius of the cylindrical surface is equal to the radius of the sphere).

4. The arc $o'6'o'_1$ is divided into equal parts: $o'1' = 1'2' = 2'3'$, etc. (in the drawing the arc $o'6'$ is divided into six equal parts).

5. Points $1'$, $2'$, etc. are taken for the vertical projections of the segments of generating elements of the cylindrical surface with the axis parallel to the line segment mn, and their horizontal projections ab, cd, etc. are constructed.

6. On a straight line passing through the points M_0 and N_0 $M_0N_0 = mn$ is laid off, and a perpendicular is drawn to the segment M_0N_0 through its midpoint.

7. The line segments $6_0O_0 = 6_0O_{10}$ respectively equal to the arcs $o'6'$ and $6'o'_1$ (i.e. $2\pi R : 4$) are laid off on this perpendicular.

8. These segments are divided into parts respectively equal to the arcs $o'1'$, $1'2'$, ..., and through the points 1_0, 2_0, etc. straight lines are drawn parallel to M_0N_0, and $A_0B_0 = ab$, $C_0D_0 = cd$, etc. are laid off on them.

9. Through the points O_0, A_0, C_0, ..., and through the points O_0, B_0, D_0, ... curves are drawn with the aid of a French curve.

As a result, an approximate development of one element of the sphere (called a gore) is obtained.

If it is required to plot a point, say S (s', s), on the development, then proceed as follows: first draw on the horizontal projection a straight line ot bisecting the gore containing the projection s, and describe an arc of a circle of radius os. Then bring the point s onto the principal meridian and find the projection s'_1. Now lay off on the development of the third gore (from its vertex) a line segment equal to the length of the arc $o's'_1$, draw through R_0 a straight line parallel to M_0N_0, and construct R_0S_0 equal to rs.

Sec. 70. Examples of Constructing the Developments of Some Particular Forms

1. The surface represented in Fig. 443 is a combination of the surfaces of a prism and an oblique circular cylinder.

To develop the surface of the cylinder we divide the semicircle into equal parts by points 1, 2, 3, ..., and pass generating elements through them. The vertical projections of these elements are equal to their segments. Now we draw through the point $1'$ the trace of a vertical projecting plane T intersecting the cylinder and thus yielding its normal section. On the line 4_04_0 we lay off segments 4_0E_0, 4_0D_0, 4_0C_0 equal to the vertical projections $4'e'$, $4'd'$, $4'c'$, and then draw through E_0, D_0, and C_0 straight lines perpendicular to the line 4_04_0. Now from 4_0 as centre we strike an arc of radius equal to the chord 4-3 to intersect the line drawn through the point C_0, thus obtaining point 3_0. From this point as centre we strike once again an arc of the same radius intersecting the line drawn through the point D_0, and

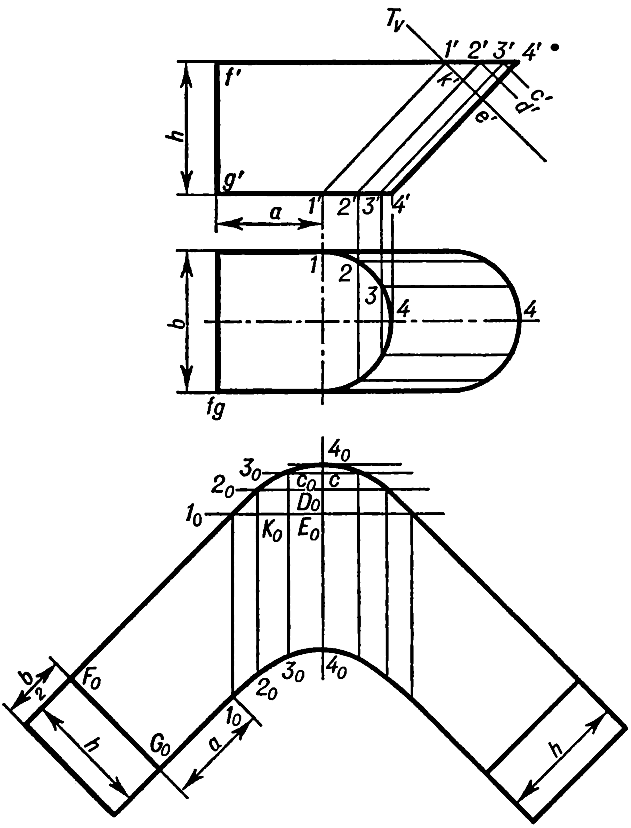


Fig. 443

from the point thus obtained (2_0) we describe an arc of the same radius to intersect the line drawn through the point E_0.

The above considered construction is based on developing the surface elements which are projected on a plane in the form of triangles. Consider one of such triangles *1'k'2'* on the *V* plane. Its leg *k'2'* represents a segment of a generating element projected without distortion, the hypotenuse *1'2'* represents the projection of an arc of the semicircle, and the leg *1'k'* is the projection of a portion of an ellipse obtained as a normal section of a given cylindrical surface. When developing the given solid, we have to construct a right triangle by its leg *2'k'* and the hypotenuse (the chord *1-2*).

On having determined the positions of points 1_0, 2_0, 3_0, we draw through them and through the point 4_0 a smooth curve which is taken for the development of the circular arc*. Drawing $1_0 1_0$, $2_0 2_0$, ..., we obtain points

*Figure 443 represents half of the development.

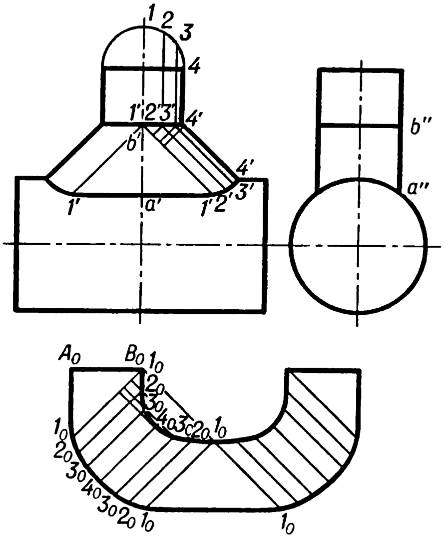


Fig. 444

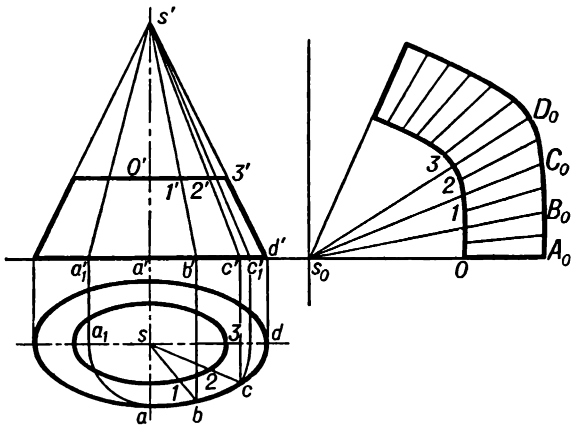


Fig. 445

for constructing the curve representing the development of the lower arc of the circle. At points 1_0 and 1_0 we draw straight lines tangent to the constructed curves. The rest of the constructions is obvious from the drawing.

2. Figure 444 shows the development of the surface of a passage coupling two cylinders. This intermediate part is bounded by the surfaces of two oblique cylinders of the same type as in Fig. 443, and two planes.

We begin developing with the line *AB*: the triangle $A_0B_0I_0$ congruent to the triangle $a'b'I'$ is constructed, and the development of the cylindrical surface is attached to it (this development is made analogously to that illustrated in Fig. 443); then the triangle $I_0I_0I_0$ is drawn congruent to the triangle $I'I'I'$, and so on.

3. Figure 445 shows the development of the lateral surface of a frustum of an elliptical cone.

On finding the vertex of the cone (s', s), we divide the upper ellipse by points *1*, *2*, ... The elements drawn from the point *S* to the points *1*, *2*, ... divide the conical surface into parts which are developed into triangles. For instance, the portion *SCD* of the conical surface is developed into the triangle $S_0C_0D_0$ in which the sides S_0D_0 and S_0C_0 are equal to the generating elements *SD* and *SC* (the length of the element *SC* is determined by the method of revolution), while the side C_0D_0 is taken as a line segment equal to the rectified arc *cd* (by dividing it into small parts).

On finding the points C_0, B_0, A_0 and those located symmetrical to them about the element S_0D_0, we draw a smooth curve representing the development of the lower ellipse. We then lay off $D_0 3$, $C_0 2$, etc. equal to the lengths of the segments of the elements *D3*, *C2*, etc. to find the curve representing the development of the upper ellipse. Given in Fig. 445 is one half of the complete development.

4. Figure 446 demonstrates the development of the lateral surface of a frustum of an oblique circular cone. The left-hand drawing shows the development constructed in the way used in Fig. 445; and the right-hand drawing illustrates another method consisting in that the given surface is replaced by a polyhedral surface inscribed in it. Using the horizontal projection of the vertex of the cone (point *s*), we first carry out the division on the horizontal projection by drawing straight lines from this point. For instance, by drawing *sa* we get the projection *ab* of a segment of the generating element. Using the points on the horizontal projection, we obtain the division of the vertical projection. Further, we consider, for instance, the plane element *ACDB*, draw the diagonal *BC* in it, and determine the lengths of segments for constructing triangles. One side of each triangle is the chord of the corresponding circle of the horizontal projection. The development is composed of such triangles; the polygonal lines are replaced by smooth curves drawn through the vertices of the polygonal lines.

5. Figure 447 shows the construction of the development of the surface of an annulus. The projection represents a bend equal to a quarter of an annulus, the development giving the surface of a third of this bend.

A straight line $o'a'$ is drawn which is the axis of symmetry of the projection of the considered portion of the bend, thus determining a circle (the normal section) whose development in the form of a straight line D_0D_0 is taken for the midline of the figure to be obtained as the development of the considered portion of the annulus. Concentric arcs are drawn on this section from point o' corresponding to the points of division *1*, *2*, ... The development is constructed separately for parts *I* and *II*. For the first part we lay

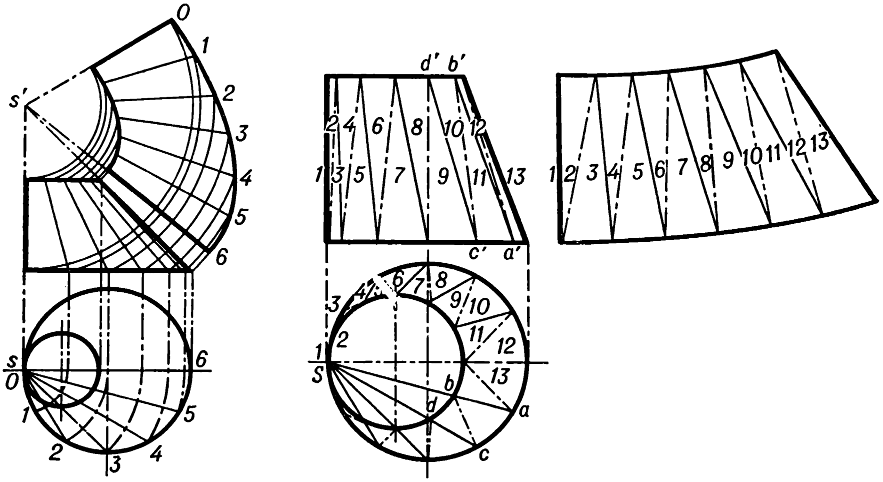


Fig. 446

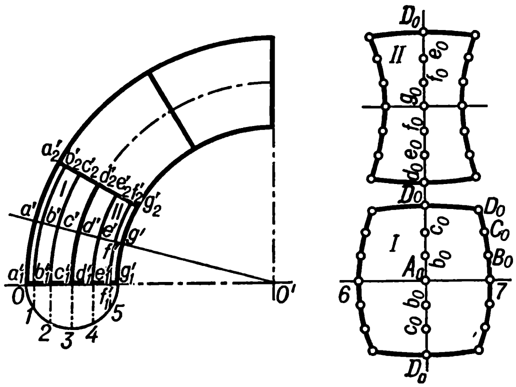


Fig. 447

off a line segment D_0D_0 equal to half the length of the circle obtained in the normal section, and divide it into parts in accordance with the initial divisions *1*, *2*, ... At point A_0 we draw a perpendicular to D_0D_0 and lay off on it (on both sides of the point A_0) line segments $A_0 6$ and $A_0 7$ equal to the arcs $a'a_1'$ and $a'a_2'$. Point B_0 is determined in the following way: from point b_0 on the development we describe an arc of radius equal to the length of the arc $b'b_2'$, and from point *7* an arc of radius equal to the length of the line segment $a_1'1$. To construct the points C_0, D_0 we proceed in the same way.

The development of the surface of part *II* of the bend is constructed analogously.

6. Figure 448 shows how a surface of revolution generated by a curvilinear generatrix is developed.

First we divide the surface into equal parts by meridians. The drawing shows the development of a sixth part of the surface.

We draw a chord $b'c'$, bisect it, and through the point of division K draw a perpendicular to intersect the arc $b'c'$. We then bisect the segment of this perpendicular from the point K to the point of intersection with the arc and through the point of division draw a straight line parallel to the chord $b'c'$. Now we divide the segment $1'7'$ into a certain number of equal parts and, through the points of division, pass horizontal planes cutting the surface of revolution in circles (i.e. parallels). We begin constructing the development with the midline (straight line $s'E_0$). On this line we lay off segments $1_0 2_0$, $2_0 3_0$, ... respectively equal to $1'2'$, $2'3'$, ... by describing arcs of radii $s'1'$, $s'2'$, ... from the point s'. From points 1_0, 2_0, ... on the arcs we finally lay off the lengths of the arcs of the horizontal projections of the parallels of the developed portion of the surface (for instance, $7_0M_0 = cm$ and $7_0N_0 = cn$).

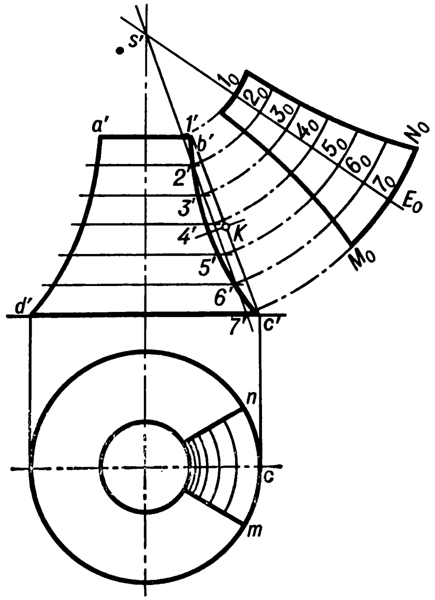


Fig. 448

QUESTIONS TO CHAPTER 11

1. What methods are applied for constructing the developments of cylindrical and conical surfaces?

2. How do we construct the development of the lateral surface of a frustum of a cone if it is impossible to complete the frustum?

3. How do we construct a conventional development of a spherical surface?

CHAPTER 12

AXONOMETRIC PROJECTIONS

Sec. 71. General

Axonometric projections* are widely used in engineering due to their pictorial force and simplicity in construction. Exercises in constructing axonometric projections of objects help a great deal in acquiring the skill of reading and understanding the language of engineering drawings, as well as in developing the ability to visualize the shapes of three-dimensional objects and to feel the proportions of machine parts.

Essentially, *the method of axonometric projection consists in the fact that an object is referred to some coordinate system and then is projected by parallel lines or rays onto a plane together with the system of coordinates***.

In mechanical engineering axonometric projections are used as an auxiliary to orthographic projections of a mechanical part when the necessity is felt to give a clearer picture of its shapes which are difficult to visualize from the orthographic projections. Without the axonometric picture it is sometimes very difficult to visualize the shape of the object from the three orthographic projections alone.

Axonometric projections differ from orthographic (orthogonal) projections in that in axonometry an object is projected only onto one plane of projection called the *axonometric* (or *picture*) *plane* and is placed in front of the picture plane so as to expose three sides to the viewer.

Figure 449 shows the scheme of projecting a point A of space referred to a system of rectangular coordinates $Oxyz$ onto a plane P taken for the

*The word 'axonometry' (Greek) consists of two words: 'axon'—axis and 'metreo'—I measure, and means 'measuring with the aid of axes', or 'measuring along axes'.

**From now on we shall call a parallel projection axonometric, bearing in mind, however, that an axonometric projection may also be a central projection.

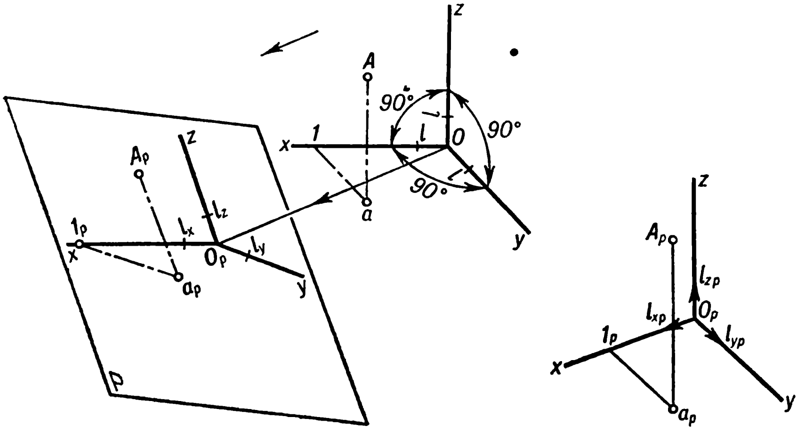


Fig. 449

plane of axonometric projections. The direction of projection is defined by an arrow*.

The straight lines Ox, Oy, Oz represent the coordinate axes in space, and O_px, O_py, O_pz their projections on the plane P. The latter are called the *axonometric axes* (or the axes of axonometric coordinates).

A line segment l is laid off on the axes x, y, z, and is taken as a unit of measurement along these axes (the *true unit*). The line segments l_x, l_y, l_z on the axonometric axes represent the projections of the segment l. In the general case they are not equal to l, and are not equal to one another. The segments l_x, l_y, l_z are called the *axonometric units* and are used for measuring along the axonometric axes**.

Distortion of the segments of the coordinate axes during projection on plane P is characterized by so-called *distortion factors* (ratios of foreshortening, or scale ratios). The distortion factor is the ratio of the length of the projection of an axis on the picture to its true length. Thus, the ratios $\frac{l_x}{l}$, $\frac{l_y}{l}$, $\frac{l_z}{l}$ are distortion factors along the axonometric axes. Let us denote the distortion factor along the axis O_px by k, along the axis O_py by m, and along the axis O_pz by n.

The three-segment space line $O1aA$ is projected into a plane polygonal line $O_p1_pa_pA_p$ (Fig. 449). The point A_p is the *axonometric projection* of the point A; the point a_p represents the axonometric projection of the point a which is one of the orthographic projections of the point A, namely, on the

*The direction of projection may form an acute or a right angle with the plane of axonometric projections. To ensure an obvious representation this direction should be parallel to none of the projection planes.

**The terms 'axonometric scales' and, accordingly, 'the true scale' are also used.

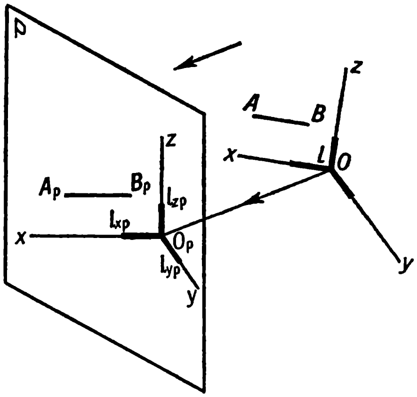


Fig. 450

H plane (xOy). The point a_p is called the *secondary projection* of the point A. This term clearly expresses the fact that point a_p is obtained by means of two succesive projections.

We may construct two more secondary projections of the point A corresponding to its two other orthographic projections on the planes V (xOz) and W (yOz).

The ratios of the axonometric projections of the line segments parallel to the rectangular axes of coordinates to the line segments themselves are expressed by the distortion factors k, m, n.

Since (see Fig. 449) al is parallel to Oy and aA is parallel to Oz, a parallel projection results in a_pl_p being parallel to O_py and a_pA_p to O_pz. The parallel projection preserves the ratio of parallel line segments, hence: $a_pl_p : l_y = al : l$ or $a_pl_p : al = l_y : l = m$, where m is the distortion factor along the axis O_py. Analogous conclusions may be drawn with respect to the line segments arranged parallel to the axes x and z: the ratios of the projections of such line segments to the segments themselves are equal to the distortion factors k and n, respectively.

For instance, the ratio of the axonometric projection A_pB_p of the line segment AB parallel to the x-axis (Fig. 450) to the line segment itself is equal to $A_pB_p : AB = k$.

Each of the segments of the line $OlaA$ defines one of the rectangular coordinates of the point A, the projections of these segments (the segments of the polygonal line $O_pl_pa_pA_p$) defining correspondingly the *axonometric coordinates* of the same point A. Obviously, with the aid of the distortion factors we can pass over from the rectangular coordinates to the axonometric, and vice versa: $x_p = kx$, $y_p = my$, $z_p = nz$, where x_p, y_p, z_p denote the line segments defining the axonometric coordinates of the point, and x, y, z its rectangular coordinates.

Figure 451 gives an example of the construction of an axonometric projection by its orthographic projections.

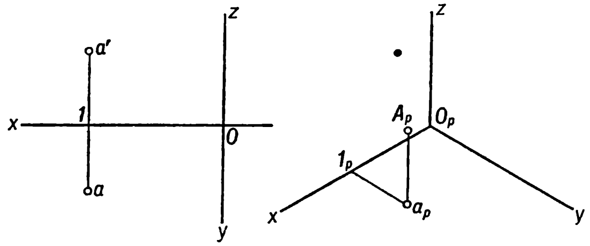


Fig. 451

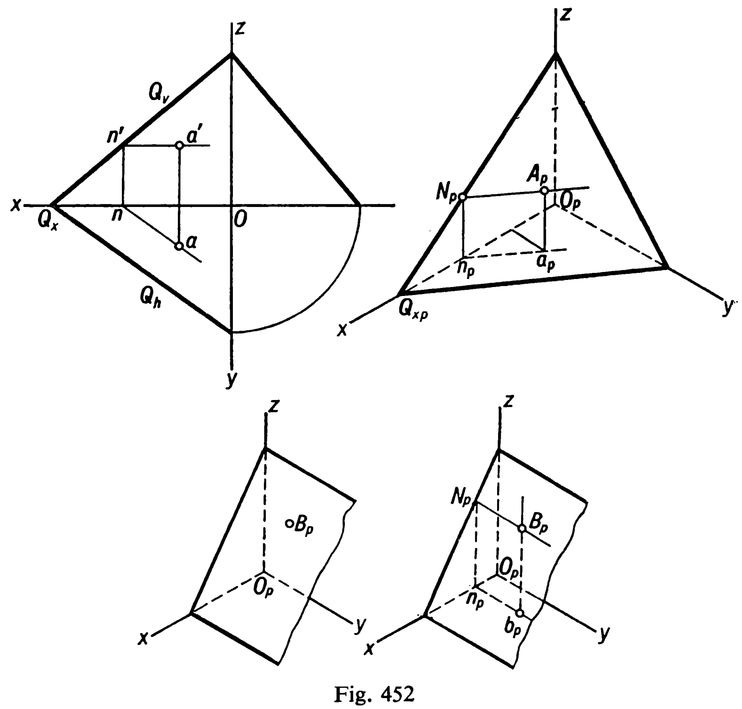


Fig. 452

The **point** A_p is constructed by the coordinate line segments taken from the drawing: $x = O1$, $y = a1$, $z = a'1$. Taking into consideration the distortion factors k, m, and n, we lay off on the axis O_px the line segment $O_p1_p = k \cdot O1$, draw the line segment $1_pa_p = m \cdot a1$ parallel to the axis O_py and, finally, the line segment $a_pA_p = n \cdot a'1$ parallel to the axis O_pz.

The **plane** Q (Fig. 452) is represented by its traces in axonometric projection. We construct the traces taking the points of their intersection with the axes by means of the corresponding intercepts (for instance, the point Q_{xp} is constructed with the aid of the intercept $OQ_x : O_pQ_{xp} = k \cdot OQ_x$).

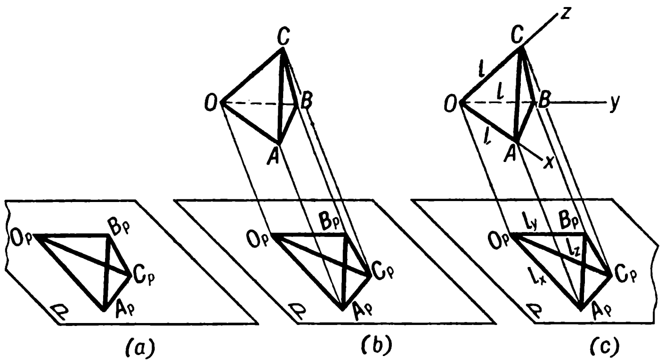


Fig. 453

Point A contained in the plane Q is constructed in axonometric projection by its coordinates. The horizontal line N_pA_p must be parallel to its secondary projection and the trace on the plane xO_py. The point A_p could also be constructed as the point of intersection of two straight lines contained in the plane Q, by constructing the axonometric projections of these lines.

The same figure represents the axonometry of a vertical projecting plane containing a point B_p. How do we determine the rectangular coordinates of this point? The construction is shown in the right-hand drawing of Fig. 452 (below): we draw (in axonometry) a horizontal line N_pB_p and construct its secondary projection to obtain the secondary projection b_p. The required coordinates of the point B are

$$x = \frac{O_pn_p}{k}, \quad y = \frac{n_pb_p}{m}, \quad z = \frac{b_pB_p}{n};$$

where k, m, n are the distortion factors.

When constructing axonometric projections use is usually made not of the distortion factors but of certain quantities proportional to them; these quantities will be called the *reduced distortion factors**.

Using the reduced factors, we may put the greatest of them equal to unity which simplifies the construction involved.

If we take on a plane P four arbitrary points O_p, A_p, B_p, and C_p of which no three are collinear, and join them pairwise by straight lines, then we get a figure called the *complete quadrilateral* ($O_pA_pB_pC_p$); *this is the quadrilateral with its diagonals* (see Fig. 453, *a*). If through these points we draw parallel straight lines and take on each of them an arbitrary point (O, A, B, C) so that all the taken points are non-coplanar, then in the general case we get a tetrahedron $OABC$**. Obviously, there is an infinite number of tetrahedrons

*The term 'reduced distortion factors' was introduced by N. F. Chetverukhin and E. A. Glazunov.

**In this case a triangular pyramid of an arbitrary shape.

in space whose parallel projection may be represented by the complete quadrilateral $O_pA_pB_pC_p$ and among them a tetrahedron with a right trihedral angle at point O and equal edges OA, OB, OC. This tetrahedron may be considered as a *scale tetrahedron**, i.e. the three equal and mutually perpendicular edges of this tetrahedron serve as scales of the coordinate axes in space (Fig. 453, *c*). This is the essence of the *basic statement of axonometry* (or "the basic theorem of axonometry") formulated as follows: *any complete quadrilateral in the plane is always a parallel projection of a scale tetrahedron.* Therefore, any three non-coincident straight lines passing through a point in the plane may be taken for the axonometric axes, i.e. for the projections of the axes of the rectangular coordinates, and any three line segments laid off on these lines from the point of their intersection may be taken for the axonometric units in conformity with the chosen ratio of the reduced distortion factors**.

If the distortion factors are equal along all three axes ($k = m = n$), then the axonometric projection is called *isometric*; if the distortion factors are equal along any two axes and if the third differs from these two (for instance, $k = n$, but m is not equal to k, or $k = m$, but n is not equal to k), then the projection is called *dimetric*; and finally, if $k \neq m$, $k \neq n$, $m \neq n$, then the projection is termed *trimetric****.

Axonometric projections likewise differ in the angle which is formed by a projecting line with the projection plane P. If this angle is not equal to 90°, the axonometric projection is *oblique*, and if it is equal to 90°, the projection is *rectangular*. Naturally, isometric, dimetric and trimetric projections may be either rectangular or oblique.

For the sake of comparison let us imagine a sphere in rectangular and oblique axonometric projections. In the first case the elements of the projecting cylindrical surface enveloping the sphere are perpendicular to the plane of axonometric projections. Since the projecting cylinder is a cylinder of revolution, a *rectangular axonometric projection of a sphere is a circle.* In the second case (an oblique projection) the intersection of the projecting surface with the plane of axonometric projections yields an ellipse. In an oblique axonometric projection the representation of the sphere is less descriptive.

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*The term 'scale tetrahedron' was introduced by N. F. Chetverukhin.

**"The basic statement of axonometry" was formulated by K. Polke (in 1851) in the form of the following theorem: *any three line segments emanating from a single point in the plane may be taken for a parallel projection of three equal and mutually perpendicular line segments in space.* In the sixties of the last century G. Schwartz generalized Polke's theorem. He proved that any complete quadrilateral in the plane may always be regarded as a parallel projection of a tetrahedron similar to any given one.

***The ancient Greek 'isos' means 'equal'; in isometric projection the distortion factors are equal along all three axes; 'di' means 'double'; in dimetric projection the distortion factors are equal only along two axes; 'treis' means 'three'; in trimetric projection the distortion factors are different along all three axes.

In the practice of constructing obvious representations use is made only of certain definite combinations of directions of the axonometric axes and the distortion factors (or reduced distortion factors).

Sec. 72. Rectangular Axonometric Projections. The Distortion Factors and Angles Between the Axes

1. Let us take the plane of axonometric projections so that it intersects all three coordinate axes at points X, Y, Z (Fig. 454, left). In the case of rectangular axonometric projections the line segment OO_p is perpendicular to the plane P. The segments O_pX, O_pY, and O_pZ (axonometric projections of the x-, y-, and z-intercepts) represent the legs of right triangles, the intercepts themselves being the hypotenuses of these triangles. Hence, $O_pX:OX = \cos\alpha$, $O_pY:OY = \cos\beta$, $O_pZ:OZ = \cos\gamma$. But these relationships represent just the distortion factors k, m, n. Consequently, $k = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$. For the line segment OO_p the cosines of the angles α_1, β_1, γ_1 (Fig. 454, right) complementary to the angles α, β, and γ are the *direction cosines*. Therefore, $\cos^2\alpha_1+\cos^2\beta_1+\cos^2\gamma_1 = 1$*. Since $\alpha = \frac{\pi}{2}-\alpha_1$, and so on, $\sin^2\alpha+\sin^2\beta+\sin^2\gamma=1$, i.e. $1-\cos^2\alpha+1-\cos^2\beta+1-\cos^2\gamma = 1$, whence $\cos^2\alpha+\cos^2\beta+\cos^2\gamma = 2$. Consequently, $k^2+m^2+n^2 = 2$, i.e. *for a rectangular axonometric projection the sum of the squares of the distortion factors is equal to two.*

2. Isometric projection.** Since $k = m = n$, we have $3k^2 = 2$, whence

$$k = \sqrt{\frac{2}{3}} \approx 0.82.$$

This means that in a rectangular isometric projection the foreshortening along all three axes (or along straight lines parallel to the axes) is approximately equal to 0.82.

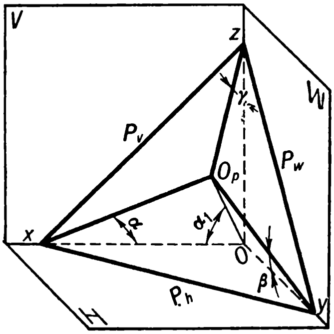
3. Dimetric projection. Two distortion factors are equal to each other, and the third is not equal to them. If we take $k = n$, and put $m = 1/2\ k$, we shall obtain

$$2k^2+1/4k^2 = 2, \quad k = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3} \approx 0.94.$$

Consequently, in a rectangular dimetric projection the foreshortening along two axes (in the present case along the axes O_px and O_pz) or along

*We recall here how this relationship is derived (Fig. 454, right): $OK^2 = OX^2+OY^2+OZ^2$, but $OX = OK\cdot\cos\alpha_1$, $OY = OK\cdot\cos\beta_1$ and $OZ = OK\cdot\cos\gamma_1$, whence $OK^2 = OK^2\cos^2\alpha_1+OK^2\cos^2\beta_1+OK^2\cos^2\gamma_1$, and (after reduction by OK^2) $1 = \cos^2\alpha_1+\cos^2\beta_1+\cos^2\gamma_1$.

**The term 'isometric projection' was first suggested by William Farich in his reports delivered in 1820 in Cambridge (England).



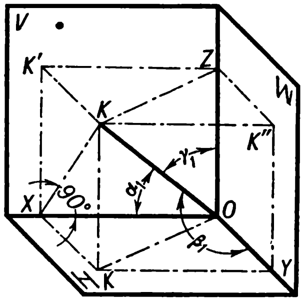


Fig. 454

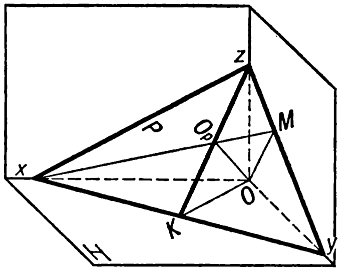


Fig. 455

straight lines parallel to these axes is approximately equal to 0.94, and along the third (that is along the axis O_py) to ≈ 0.47.

4. Intersecting the coordinate planes, the plane of axonometric projections forms a triangle called the *triangle of traces*.

Let us prove that *in rectangular axonometric projections the axonometric axes are the altitudes of the triangle of traces.*

Indeed, if OO_p is perpendicular to P, then OK is perpendicular to XY and, by virtue of the theorem on three perpendiculars, ZK is perpendicular to XY. (Fig. 455). Analogously, XM is perpendicular to YZ. The point O_p is the point of intersection of the altitudes (the *orthocentre*) of the triangle of traces.

Furthermore, *in rectangular axonometric projections the triangle of traces is an acute-angled triangle.*

Indeed, in this case the orthocentre is located inside this triangle, and such position of the orthocentre is characteristic only of an acute-angled triangle.

It follows from this fact that the angles XO_pZ, XO_pY, and YO_pZ are obtuse. Indeed, since the triangle of traces is an acute-angled one, the angle between the altitudes supplements the acute angle up to 180°, for instance,

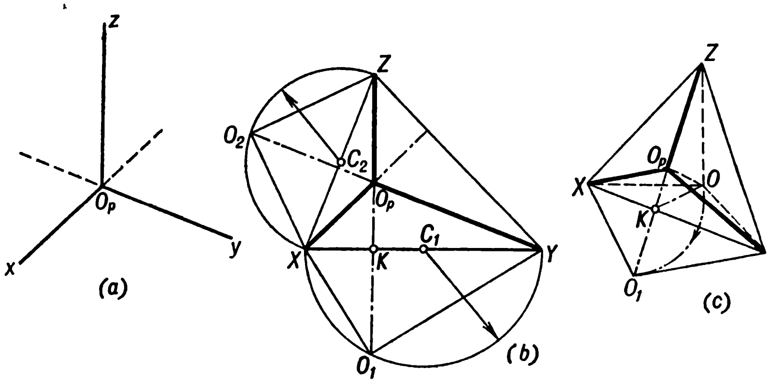


Fig. 456

$\angle MO_pK = 180° - \angle XO_pK$; but $\angle XO_pK$ is an acute one, hence, $\angle MO_pK$ is obtuse.

But it does not follow from this fact that in a rectangular axonometric projection we may use only such arrangement of the axes which is shown in Fig. 456, *a*. Let the x- and y-axes be extended beyond the point O_p as it is indicated in the figure. In such a case the angle between the extended axes will remain obtuse, but the angles formed by either of these axes with the z-axis will turn out to be acute. But it is not difficult to establish that in a rectangular axonometric projection the choice of the axes is still limited, and namely, *it is necessary that an obtuse angle between two axes be divided by the extension of the third axis, and an acute angle between two axes could not be divided by the extension of the third axis.*

5. Suppose we are given the axes for a rectangular axonometric projection (Fig. 456, *a*) and it is required *to determine the distortion factors for the given arrangement of the axes.*

First of all we construct a triangle whose altitudes are directed parallel to the given axes (Fig. 456, *b*). This triangle serves as the triangle of traces in which the angle XO_pY is obtained as the projection of a right angle between the axes x and y in space. Let us bring both angles XO_pY and XOY into coincidence with the plane of the drawing by revolving the angle XOY about the straight line XY (Fig. 456, *c*). Figure 456, *b* shows that bisecting XY with point C_1 and describing a semicircle of radius C_1X from this point, we can project the point O_p along a perpendicular to XY onto the semicircle. The point O_1 is the vertex of the right angle between the x- and y-axes in space after the revolution.

Now the distortion factors are determined from the relationships $O_pX : O_1X = k$ and $OY : O_1Y = m$. To determine the factor n, we may use the formula $k^2+m^2+n^2 = 2$ or to construct a semicircle on XZ as the diameter, and to take the ratio $O_pX : O_2X = n$.

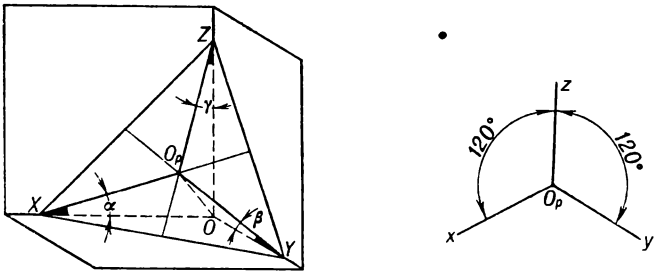


Fig. 457

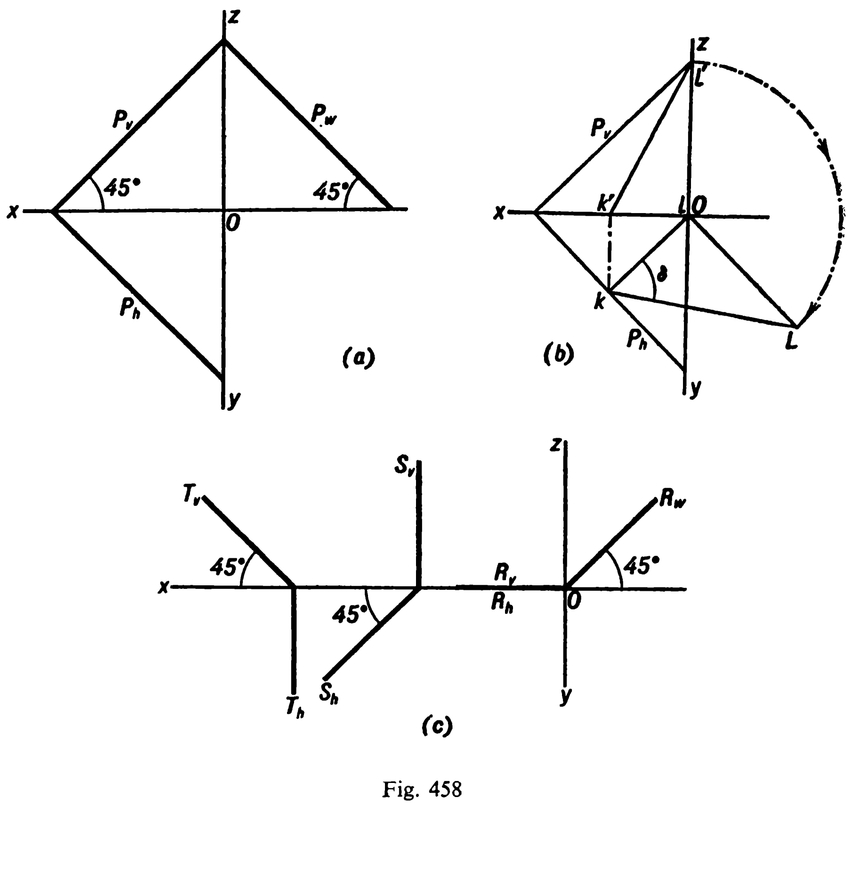


Fig. 458

6. In items 2 and 3 of this section we evaluated the distortion factors for rectangular isometric and dimetric projections. *Using these values of the distortion factors, we can determine the angles between the axes for these rectangular axonometric projections from the triangles of traces*.*

Isometric projection (Fig. 457). Since we consider a rectangular projection, the straight line OO_p is perpendicular to the plane containing the triangle of traces.

In isometric projection the distortion factors are equal along all three axes: $k = m = n$; consequently, $\cos\alpha = \cos\beta = \cos\gamma$ and $\alpha = \beta = \gamma$ (the angles are acute).

It follows from this fact that for isometric projection the triangle of traces is *isosceles*. Hence, in the triangle of traces each of the angles XO_pZ, XO_pY, YO_pZ is equal to 120°.

Thus, for isometric projection we obtain the arrangement of the axes as is shown in the right-hand picture of Fig. 457.

The plane of isometric projections intersecting the positive semi-axes x, y, z will be represented in the system of orthographic projections as is shown in Fig. 458, *a*. This plane forms with each of the coordinate planes an angle $\delta \approx 55°$ (more accurate, 54°45′).

Obviously, planes arranged like those shown in Fig. 458, *c* as well as the figures contained in them will be represented in isometric projection in the form of a straight line.

Dimetric projection. Here of the three distortion factors two are equal to each other. We are going to consider the case when $k = n$, $k = 2m$. In this case the angle between the axonometric axes O_pz and O_py must be equal to 131°25′, and the axis O_px forms an angle of 7°10′ with the perpendicular to the axis O_pz.

Let us prove this. Suppose $k = n$ and, hence, $\alpha = \gamma$ and $OX = OZ$ (Fig. 457, left). Taking the line segment OX for unity, we get $XZ\sqrt{2}$ Considering the dimetric projection in which $k = n = \frac{2\sqrt{2}}{3}$ and $m = \frac{\sqrt{2}}{3}$, we may write $O_pX = O_pZ = \frac{2\sqrt{2}}{3}$. Since $OX = OZ$, we have $XY = ZY$, i.e. in this case the triangle XYZ is an isosceles one.

In this triangle (Fig. 459) the altitude YK bisects the side XZ, i.e.

$$XK = KZ = \frac{XZ}{2} = \frac{\sqrt{2}}{2}.$$

---

*More precisely, from triangles *similar* to the triangles of traces. Generally, given the distortion factors, we may construct the axes in a rectangular axonometric projection applying Weissbach's theorem: "In a rectangular axonometric projection the axonometric axes are the bisectors of the angles of the triangle whose sides are proportional to the squares of the distortion factors".

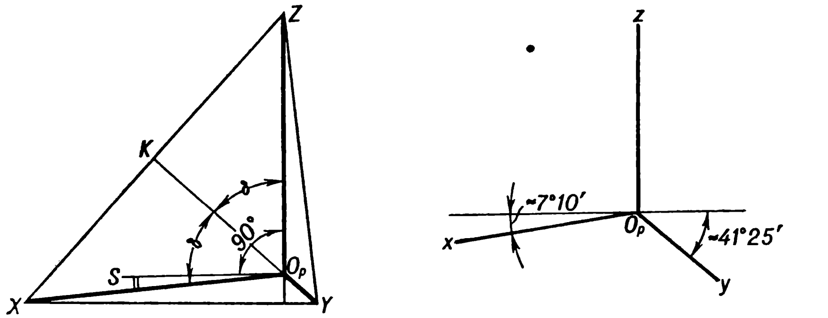


Fig. 459

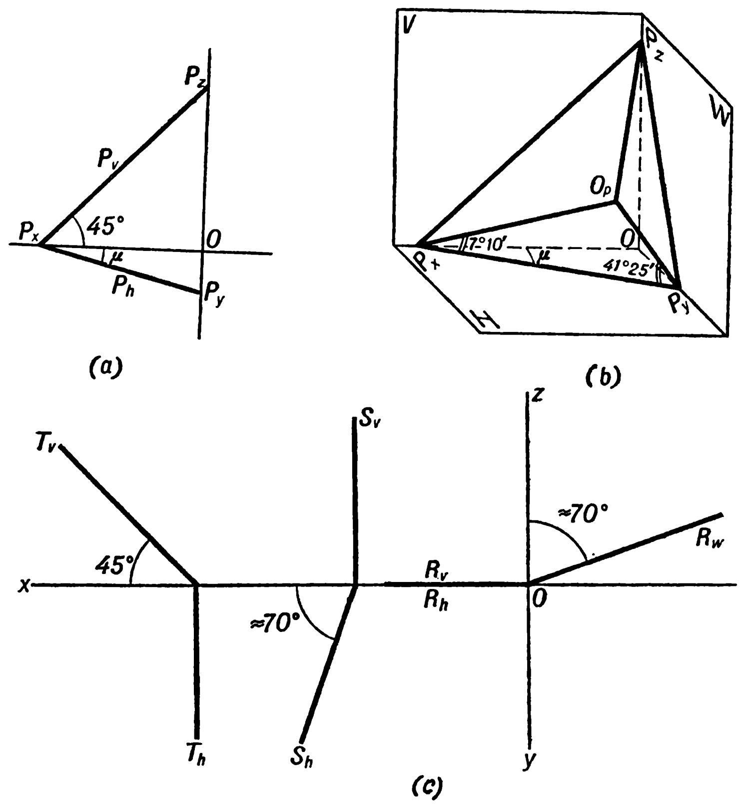


Fig. 460

From the examination of a right-angled triangle O_pKZ it follows that

$$\sin\delta = \frac{ZK}{O_pZ} = \frac{\sqrt{2}}{2} : \frac{2\sqrt{2}}{3} = 0.75.$$

The angle $\delta \approx 48°35'$; $2\delta = 97°10'$. It is seen from the figure that the angle $SO_pX \approx 7°10'$, since O_pS is perpendicular to O_pZ.

We then note that

$$\angle KO_pS \approx 48°35' - 7°10' = 41°25'.$$

And so, we have obtained the arrangement of the axes for the dimetric projection shown in Fig. 459, right, in which the distortion factors form the ratio $1 : 0.5 : 1$.

We may construct the axis O_px, taking tan $7°10'$ equal to $1/8$, and the axis O_py, taking tan $41°25'$ equal to $7/8$. *The axis O_py can be drawn in another way, i.e. as an extension of the bisector of the angle zO_px* (see Fig. 459, left). *This method is preferable.*

If the plane of the dimetric projection under consideration, intersecting the positive semi-axes x, y, z, is represented in the system of orthographic projections, then we shall obtain a drawing shown in Fig. 460, *a*, the angle μ being approximately equal to $20°40'$ ($OP_y : OP_x = \tan\mu = 0.377$).

Thus, if the plane of dimetric projection is to be represented in the system of orthogonal projection, we have to lay off $OP_z = OP_x$ and $OP_y \approx 0.377 \cdot OP_x$ or, when rounded off, $0.4 \cdot OP_x$.

Obviously, planes arranged like those shown in Fig. 460, *c* and the figures contained in them will be represented in dimetric projection in the form of a straight line.

Line segments located parallel to the coordinate axes in space are foreshortened when projected in axonometry to an extent expressed by the corresponding distortion factors. But among line segments located in space there are such segments whose lengths do not change in axonometric projection. These are segments located in space parallel to any of the sides of the triangle of traces. Indeed, any line segment arranged, for instance, parallel to the trace XY (Fig. 457, left), the segment XY included, preserves its length in axonometric projection. But in a rectangular axonometric projection these segments turn out to be arranged perpendicular to the axonometric axes as straight lines parallel to the sides of the triangle of traces.

We shall confine ourselves to considering the two above mentioned rectangular axonometric projections—isometric and dimetric with the ratio of the distortion factors $1 : 0.5 : 1$ and the axes arranged as in Fig. 459. Later on, when using the names 'isometric' and 'dimetric' projections, we shall mean just these rectangular axonometric projections.

In constructing the considered projections the following deviations are allowed:

(1) in isometric projection the distortion factors $\sqrt{\frac{2}{3}}$ (≈ 0.82) are mostly not used; they are replaced by the reduced factors equal to unity;

(2) in dimetric projection the *distortion factors* $\frac{2\sqrt{2}}{3}$ (≈ 0.94) and $\frac{\sqrt{2}}{3}$ (≈ 0.47) *are usually not used*; they are replaced by the reduced factors 1 and 0.5, respectively.

The replacement of the true factors by more convenient numbers (that is, by reduced factors) is of considerable importance for the construction practice. Somewhat enlarged representations resulting from this replacement may turn out to be unacceptable only in special cases of construction. In the latter cases the true distortion factors should be used.

The elongation of line segments in an isometric projection constructed with the use of the reduced distortion factors is expressed by the ratio $1 : \sqrt{\frac{2}{3}} \approx 1.22$, and in dimetric projection by the ratio $1 : \frac{2\sqrt{2}}{3} \approx 1.06$.

For instance, the line segments parallel in space to the sides of the traingle of traces and, consequently, laid off in axonometric projection in the directions perpendicular to the axonometric axes are elongated by 1.22 times as compared with the true length in isometric projection, and by 1.06 times in dimetry.

Sec. 73. Constructing a Rectangular Axonometric Projection of a Circle

1. Let us begin with the general problem: *Construct a rectangular axonometric projection of a circle contained in an oblique plane Q.*

If the plane Q forms an acute angle φ with the plane of axonometric projections P, then the axonometric projection of the circle is an *ellipse* (Fig. 461). The major axis of this ellipse is the projection of the circle's diameter which is parallel to the line MN of intersection of the plane Q and P, the minor axis of the ellipse being the projection of the circle's diameter which is arranged perpendicular to MN, i.e. located on the line determining the inclination of the plane Q with respect to the plane P. If C is the centre of the circle contained in the plane Q, then the minor axis of the ellipse obtained in projecting this circle into the plane P will be found on the line C_pK. The length of the minor axis of the ellipse depends on the magnitude of the angle φ between the planes Q and P; if (Fig. 462) the line segment CB is equal to the radius (R) of the circle, then the semiminor axis of the ellipse $C_pB_p = R\cos\varphi$.

2. If $\varphi = 0°$, then $C_pB_p = R$ which means that the plane Q_1 (Fig. 463) is parallel to the plane of axonometric projections P, and the axonometric projection of the circle contained in the plane Q_1 represents a *circle*.

If $\varphi = 90°$, then $C_pB_p = 0$: the plane Q_2 (Fig. 463) is perpendicular to the plane of axonometric projections P, and the axonometric projection of the circle contained in the plane Q_2 represents a *line segment*.

If the circle is projected into an ellipse, we may construct the projections of any two mutually perpendicular diameters. The two conjugate diameters obtained enable us to construct the ellipse itself and also to find its axes.

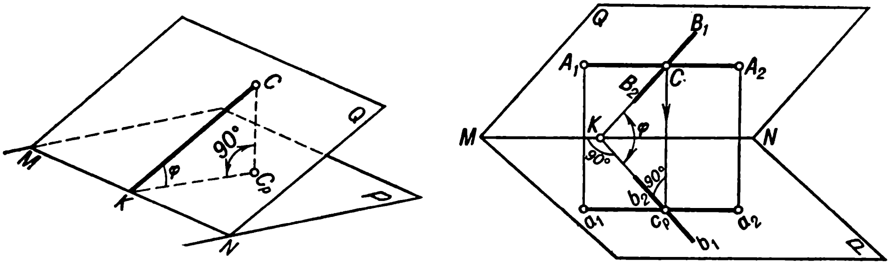


Fig. 461

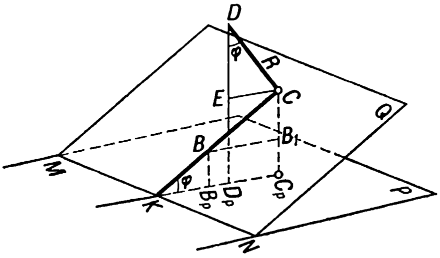
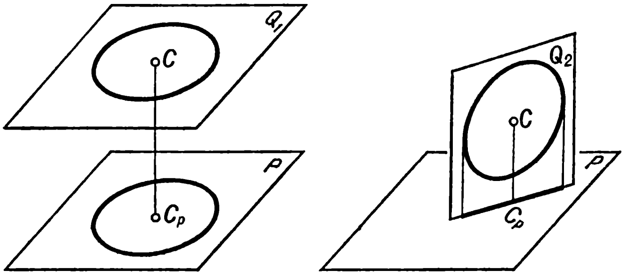


Fig. 462



(a) (b)

Fig. 463

3. Considered below is the direct construction of the axes of the ellipse (i.e. a rectangular axonometric projection of a circle) which is reduced to finding the direction and length of its minor axis.

Since the length of the minor axis depends only on the length of the diameter of the projected circle and the magnitude of the angle φ (see above), obviously, in many cases we shall obtain ellipses with repeated lengths of the axes. To get this result, it is necessary and sufficient that all circles be of one and the same diameter and are contained in planes forming equal angles with the plane of axonometric projections.

Such planes are tangent to a cone of revolution whose axis is perpendicular to the plane of axonometric projections and the generatrix makes an angle φ with this plane. Let us call this cone a directing one.

For example, circles contained in horizontal, vertical, and profile planes are represented in isometry in the form of ellipses whose minor axis is approximately equal to 0.58 of the length of the major axis (see below). But if we take a circle in a plane inclined to the plane of isometric projections at an angle of $\approx 54°45'$, i.e. at an angle formed by the planes H, V, W with the plane of isometric projections, then the ratio of the length of the minor axis of the ellipse (the isometric projection of a circle) to the length of its major axis will also be equal to ≈ 0.58.

Let us imagine a rectangular tetrahedron formed by the projection planes and the plane of isometric projections with a directing cone placed in it. The vertex of the cone is located at point O, the circumference of its base turns out to be inscribed in the triangle of traces, and the generatrix forms an angle $\varphi \approx 54°45'$ $(\tan\varphi = \sqrt{2})$ with the plane of isometric projections. The circles contained in the planes tangent to the directing cone are represented in isometric projection by ellipses whose minor axis constitutes approximately 0.58 of the length of the major axis.

Thus, many congruent ellipses representing the axonometric projections of circles of one and the same diameter are obtained in many positions relative to the axonometric axes.

But we can obtain ellipses repeated not only in size but also in position relative to the axonometric axes, i.e. it is possible to get equal and equally directed projections (ellipses), though the originals (circles) are located in planes not parallel to one another. If we imagine two equal directing cones placed on the plane of axonometric projections on both of its sides and consider the planes tangent to the directing cones and having a common trace of the plane of axonometric projections (or planes parallel to them) then the circles of equal diameters located in these planes will be represented in axonometric projection by congruent and equally directed ellipses.

---

4. Let us pass over to considering the method of construction of the minor axis of the ellipse representing a rectangular axonometric projection of a circle of radius R located in a plane Q inclined to the plane of axonometric projections P at an acute angle φ. Suppose that at point C (Fig. 462) a perpendicular CD is erected to the plane Q. The projection of this perpendicular on the plane P will be situated on the same straight line C_pK on which is situated the minor axis of the ellipse representing the axonometric projection of the circle described in the plane Q from C as centre.

Consequently, the *projection of a perpendicular dropped to the plane Q onto the plane P determines the direction of the minor axis of the ellipse.*

If a line segment $CD = R$ is laid off from C on this perpendicular and a right triangle CED is constructed, then we can establish that $\Delta CED = \Delta CB_1B$ and the leg $DE = BB_1 = C_pB_p = R\cos\varphi$, i.e. equal to half the

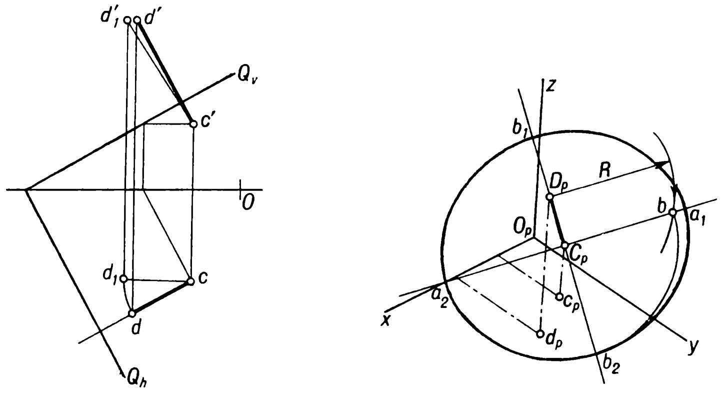


Fig. 464

length of the minor axis of the ellipse. The second leg of this triangle (the leg CE) is equal to C_pD_p, i.e. to the projection of the line segment CD on the plane P of axonometric projections.

Hence, we may construct the axes of the ellipse representing an axonometric projection of a circle of radius R contained in an oblique plane Q in the following way:

(a) to drop a perpendicular to the plane Q (in the left-hand drawing of Fig. 464) from the centre of the circle (point C) and lay off on it a line segment $CD = R$;

(b) to construct (in the given system of axonometric axes), using the coordinates of the points C and D, the axonometric projection of the segment CD, i.e. the segment C_pD_p (Fig. 464, right) which will indicate the direction of the minor axis of the ellipse;

(c) to determine the length of the semiminor axis of the ellipse which is done by erecting a perpendicular to C_pD_p at point C_p, intersecting it by a circular arc of radius R described from the point D_p as centre, and laying off the length of the obtained segment C_pb, equal to $R \cos \varphi$, on the line C_pD_p on both sides of C_p.

Thus, the minor axis of the ellipse is determined ($b_1b_2 = 2R \cos \varphi$);

(d) to lay off the segment C_pa_1 and C_pa_2, each one equal to the radius R of the projected circle, on the perpendicular erected to the line C_pD_p at point C_p; in this way the major axis of the ellipse is obtained ($a_1a_2 = 2R$).

Now the ellipse can be constructed by its axes*.

5. Let us apply the above method for constructing the axes of the ellipse representing a rectangular axonometric projection of a circle also to

*The isometric projections shown in Fig. 464 are constructed using the true distortion factors $\left(\sqrt{\frac{2}{3}} \approx 0.82\right)$.

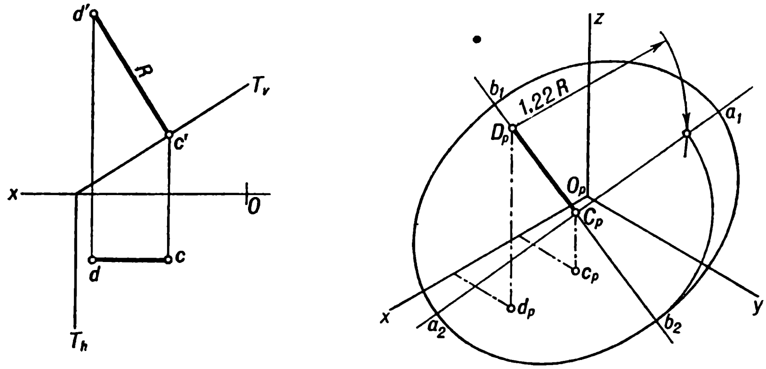


Fig. 465

the cases when the *circle is located in a projecting plane.* Here we need not construct the projections of a line segment by its length R: if the circle is contained in the plane T (Fig. 465), then any perpendicular to this plane is parallel to the V plane, and, hence, the projection obtained on this plane is equal to the projected line segment R.

The construction is carried out for two positions: in Fig. 465 a circle of radius R is contained in a vertical projecting plane T, while in Fig. 466* in a horizontal projecting plane S. The same as in the case of an oblique plane, we have to construct (using the coordinates of the points C—the centre of the projected circle—and D) the axonometric projection of the line segment CD equal to R, to determine the length of the semiminor axis, using the construction shown in Fig. 464, and, finally, to construct the ellipse by the axes thus found.

6. Let us apply the above considered method of construction to the case frequently met in everyday practice when a circle is contained *in a plane parallel to the plane of projection.* Suppose the circle is located in a horizontal plane S (Fig. 467). In this case the perpendicular dropped from the centre of the circle onto the plane S will be parallel to the z-axis and its axonometric projection (line segment D_pC_p) will be arranged parallel to the axonometric axis O_pz. But, as we know, the axonometric projection of this perpendicular defines the direction of the minor axis of the ellipse. Hence, in this case the minor axis of the ellipse turns out to be parallel to the axis O_pz and the major axis is perpendicular to this axis. Obviously, the consideration of the cases when circles are contained in vertical and profile planes will lead us to a conclusion that the major axis of the ellipse will be perpendicular to the axis O_py in the first case, and to the axis O_px in the

*The constructions shown in Fig. 465 are carried out in isometry using the reduced distortion factors, therefore $1.22R$ is taken in the drawing. Fig. 466 illustrates the constructions in dimetry using the reduced distortion factors, therefore $1.06R$ is taken.

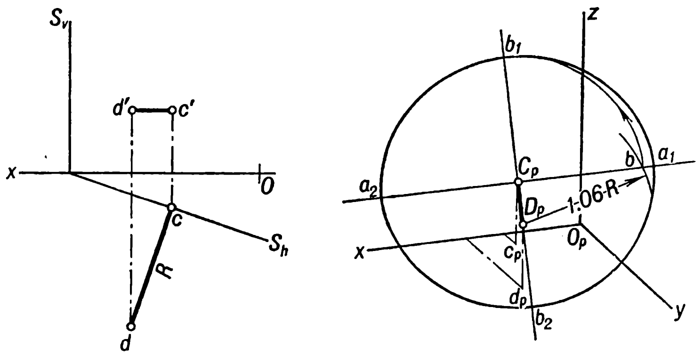


Fig. 466

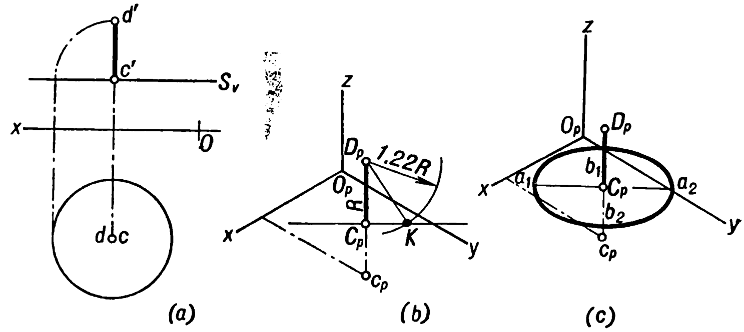


Fig. 467

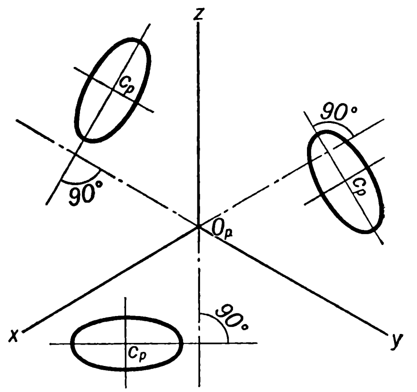


Fig. 468

second. And so, we obtain a scheme of arrangement of the axes of ellipses (see Fig. 468) in a rectangular axonometric projection of circles contained in planes parallel to the projection planes.

In these cases the length of the semiminor axis may be determined in the above manner. Once the axes are constructed, we pass over to constructing the ellipses.

We are going to apply this to the above considered isometric and dimetric projections.

7. Isometric projection. Since the plane of isometric projection is inclined to the projection planes H, V and W at one and the same angle, it is sufficient to determine the minor axis of the ellipse, say, for the case when a circle of radius R is situated in a plane parallel to the H plane.

Suppose the coordinates were laid off without multiplying them by 0.82. In this case C_pD_p (Fig. 467, *b* and *c*) becomes equal to R, and we have to strike an arc of radius $1.22R$ from point D_p as centre intersecting the perpendicular to C_pD_p. From the right triangle C_pD_pK we get: C_pK (the semiminor axis of the ellipse) $\approx \sqrt{(1.22R)^2-R^2} \approx 0.7R$. The corresponding semimajor axis will be equal to $1.22R$.

If the coordinate are laid off on being multiplied by the distortion factor 0.82, then the corresponding major axis will be equal to R and the minor axis to $0.58R$.

Thus, if a circle of diameter D is contained in a horizontal, vertical or profile plane, then in isometry the major axis of the ellipse is equal to D, and the minor axis to $0.58D$. *If we are going to use an isometric projection with reduced factors, then the axes of the above mentioned ellipses should be taken respectively equal to* $1.22D$ *and* $0.7D$.

To the four points representing the end-points of the ellipse we may add four more points, namely, the end-points of two conjugate diameters of the ellipse which are respectively parallel to two of the axonometric axes (depending on which plane of coordinates the plane containing the considered circle is parallel to). With the above mentioned enlargement (1.22) these conjugate diameters are equal to the diameter of the circle under consideration.

Suppose, for instance, it is required to construct an isometric projection of a circle of diameter 100 mm located in space in a plane parallel to the W plane. The position of the ellipse is determined by the axes O_py and O_pz. Taking in the drawing (according to a certain condition) a centre C_p (Fig. 469), we draw:

(a) a straight line perpendicular to the x-axis and lay off on it the major axis of the ellipse $a_1a_2 = 122$ mm;

(b) a straight line parallel to the x-axis and lay off on it the minor axis of the ellipse $b_1b_2 = 70$ mm;

(c) a straight line parallel to the y-axis and lay off on it the diameter of the ellipse $d_1d_2 = 100$ mm;

(d) a straight line parallel to the z-axis and lay off on it the diameter of the ellipse $e_1e_2 = 100$ mm.

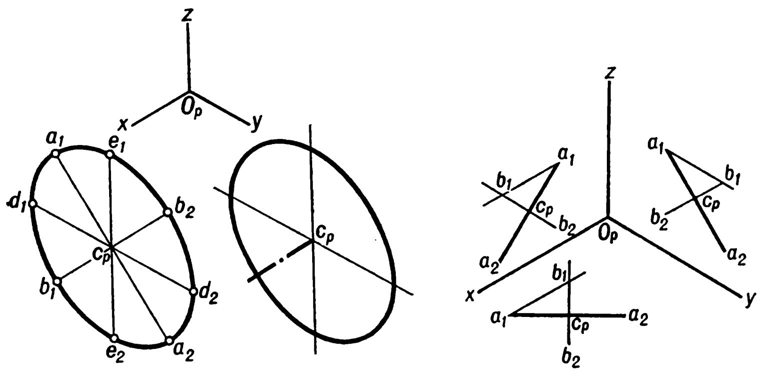


Fig. 469

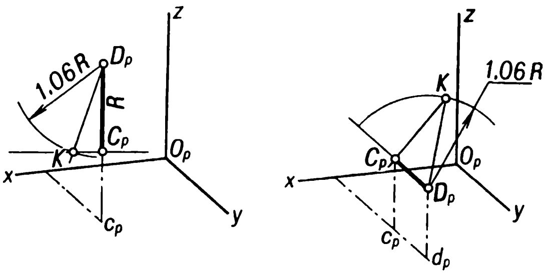


Fig. 470

The eight points thus found enable us to sketch the required ellipse freehand. When finishing the ellipse, its axes are usually erased. The usual practice is to leave only the directions parallel to the axonometric axes, one of them, namely, the direction corresponding to the axis perpendicular to the plane containing the circle is marked with a thick dot-and-dash line.

The length of the minor axis may be obtained by the method indicated in Fig. 469 (right): on constructing the major axis of the ellipse a_1a_2 and a perpendicular to it at the centre c_p of the ellipse, we draw from the end-point of the major axis (for instance, from a_1) a straight line parallel to the x-, or y-, or z-axis to intersect this perpendicular. The line segment c_pb_1 thus obtained determines the semiminor axis.

8. Dimetric projection. Since the plane of dimetric projection is inclined at one and the same angle only to two projection planes (H and W), we have to determine the semiminor axis of the ellipse for the case when the circles are contained in planes parallel to the projection planes H and W, and separately for the case when the circle is located in a plane parallel to the V plane.

Using the construction analogous to that shown in Fig. 467, we get (Fig. 470) C_pD_p parallel to the z-axis in one case, and C_pD_p parallel to the

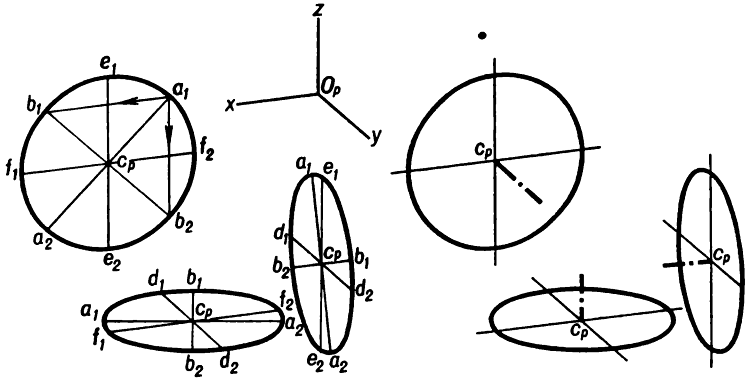


Fig. 471

y-axis in the other. Hence, in the first case $C_pD_p = R$, and in the second case $C_pD_p = 0.5R$, where R is the radius of the circle represented in dimetric projection (it should be borne in mind that dimetric projection is constructed using the reduced distortion factors 1 : 0.5 : 1).

From the right-angled triangles C_pD_pK (Fig. 470) it follows that in the first case C_pK (the semiminor axis of the ellipse) is equal to

$$\sqrt{(1.06R)^2-R^2} \approx 0.35R,$$

and in the second case it is equal to

$$\sqrt{(1.06R)^2-(0.5R)^2} \approx 0.94R.$$

Thus, if circles of diameter D are located in a horizontal or profile planes (or parallel to them), then in dimetric projection the major axis of the ellipse turns out to be equal to D, and the minor axis to $\frac{D}{3}$.

And if a circle of diameter D is situated in the vertical plane (or parallel to it), then in dimetric projection of this circle the axes of the ellipse are equal to D and $0.88D$, respectively.

But since the dimetric projection is constructed by the reduced distortion factors, *for the circles contained in the horizontal and profile planes (or in planes parallel to them) the axes should be taken equal to* $1.06D$ and $0.35D$, *and for those lying in the vertical plane (or in planes parallel to it) to* 1.06 *and* $0.94D$.

Figure 471 illustrates the construction of eight points for each ellipse in dimetry. In all the cases the axis $a_1a_2 = 1.06D$, diameters $f_1f_2 = e_1e_2 = D$, diameter $d_1d_2 = 0.5D$; as to the minor axis b_1b_2: in two positions it is equal to $0.35D$, and in one position (when it is parallel to the y-axis) to $0.94D$.

When finishing the ellipse, we indicate only the directions parallel to the

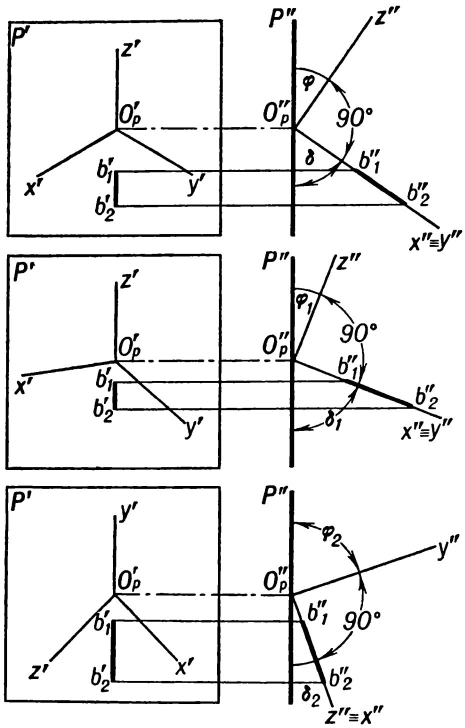


Fig. 472

axes (see the right-hand ellipses in Fig. 471). For an ellipse whose minor axis is parallel to the y-axis we can find the point b_1 by drawing from point a_1 a straight line parallel to the x-axis (if through the point a_1 a straight line is drawn parallel to the z-axis, then we get the point b_2).

9. Let us consider another method of deriving the values of the distortion factors for determining the length of the minor axis of the ellipse representing a circle located in space in the coordinate plane xOy, or xOz, or yOz (or parallel to these planes). Figure 472 represents the planes of axonometric projections coincident with the plane of the drawing, i.e. in the vertical position: (1) the plane of isometric projections, (2) the plane of dimetric projections (1 : 0.5 : 1), (3) the same, but with the y-axis in the vertical position. All the cones are supplied with the planes of axonometric projections (P'') and the coordinate axes in their position relative to the plane of axonometric projections for isometry and dimetry represented on an additional profile plane.

Since in isometric projection the angles between the coordinate axes Ox, Oy, and Oz and the plane of isometric projections are equal, and the distortion factor is equal to $\sqrt{\frac{2}{3}}$* in all three cases, the construction of the projection $O_p''z''$ is reduced to the construction of the angle φ by the value of its cosine: $\cos\varphi = \sqrt{\frac{2}{3}}$. Since the axis Oz lies in space in a profile plane, the profile projection of the coordinate plane xOy will represent a straight line at an angle of 90° to $O_p''z''$.

Now we can pass over to the computation of the factor for determining the length of the minor axis of the ellipse required for constructing an isometric projection of a circle referred to the coordinate plane xOy. Of all diameters of the circle the one inclined at an angle δ to the plane of isometric projections will turn out to be the most foreshortened. Let this diameter have the projections $b_1'b_2'$ and $b_1''b_2''$, the latter being equal to the diameter of the circle (taking into account the scale of the drawing).

Since $\delta+\varphi = 90°$, we have $\cos\varphi = \sqrt{\frac{2}{3}} = \sin\delta$. But to determine $b_1'b_2'$ from $b_1''b_2''$, we must have

$$\cos\delta = \sqrt{1-\sin^2\delta} = \sqrt{1-\frac{2}{3}} = \sqrt{\frac{1}{3}} \approx 0.58.$$

Thus, for computing the length of the minor axis of the isometric ellipse by the length of the diameter of the circle, we must take a distortion factor equal to 0.58, or a reduced distortion factor equal to 0.7. This is valid for all three cases, that is when a circle in space is located in a horizontal, or vertical, or profile plane.

Passing over now to the dimetric projection (the second and third positions in Fig. 472), we should pay attention to the fact that the plane of dimetric projections is inclined at one and the same angle only to two coordinate axes, i.e. to Ox and Oz. Therefore, two positions (second and third) are given: in the former a circle is considered in the plane xOy (the same is also true for the case when a circle is situated in the plane yOz), in the latter the circle is considered in the plane xOz.

Taking $\cos\varphi_1 = \frac{2\sqrt{2}}{3}$ for the second position and $\cos\varphi_2 = \frac{\sqrt{2}}{3}$ for the third, we get:

$$\cos\delta_1 = \sqrt{1-\frac{8}{9}} = \sqrt{\frac{1}{9}} \approx 0.33$$

and

$$\cos\delta_2 = \sqrt{1-\frac{2}{9}} = \frac{\sqrt{7}}{3} \approx 0.88,$$

for the reduced distortion factors the respective values being ≈ 0.35 and ≈ 0.94.

*All calculations are given in true factors, not in reduced one.

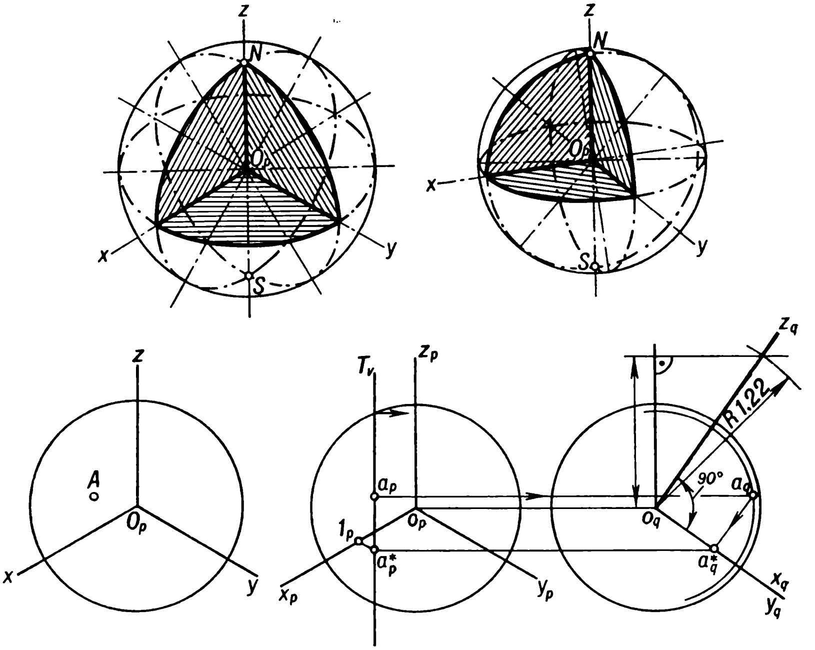


Fig. 473

Sec. 74. Examples of Constructions in Isometry and Dimetry

Given below are some examples of constructions carried out in rectangular isometric and dimetric projections.

1. Projecting a sphere. Figure 473 (top) shows a sphere represented in isometric and dimetric projections.

In both cases the sphere is shown with half-a-quarter cut away. The circles representing the contours of the projections are drawn in the following way: for the isometric projection with the radius equal to $1.22R$, and for the dimetric projection with the radius of $1.06R$, where R is the radius of the sphere. The ellipses used in both cases correspond to the equatorial and two meridional sections.

Figure 473 (bottom, left) shows a sphere represented in isometry with a point A on its visible side. The right-hand drawing demonstrates the construction of a secondary projection a_p^* (see Fig. 449) and a three-segment coordinate polygonal line $a_p a_p^* 1_p O_p$ which enables us to determine rectangular coordinates of the point A in space. The construction is carried out in supposition that the plane of isometric projections is in a vertical position, and that the axes x, y, z of rectangular coordinates inclined to it at equal angles are projected not only on this plane, but also on an additional profile plane Q.

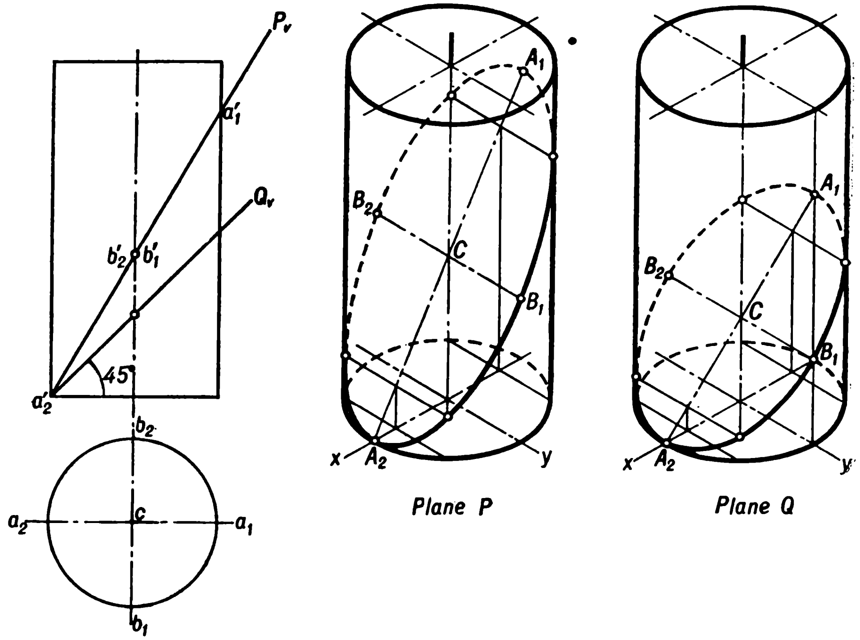


Fig. 474

Thus we get a system of projection planes P, Q and the projections a_p and a_q of the given point A. In this case a_q is obtained by cutting the sphere with a plane T. The secondary projection of the point A is also represented by two projections: a_q^* and a_p^*.

2. Lines of intersection of a cylinder and a cone with a plane. Figures 474 and 475 show the construction of isometric projections of the lines of intersection of a cylinder and a cone by vertical projecting planes*. In the cases under consideration the lines of intersection represent ellipses.

First of all we construct the lines of inclination of planes P and Q, using for this purpose the given drawing and the coordinates of the points A_1 and A_2. To construct the points belonging to the ellipses, we take auxiliary cutting planes: parallel to its generatrix and the plane yOz for the cylinder, and passing through the vertex and parallel to the y-axis for the cone. These planes are specified by their traces parallel to the y-axis on the planes of the bases of the cylinder and cone.

With the auxiliary planes chosen in such a fashion the straight lines along which they intersect with the planes P and Q turn out to be parallel to the y-axis. The intersection of these lines with the elements of the cylinder and cone yields the points belonging to the required ellipses.

*The construction is carried out using the reduced distortion factors.

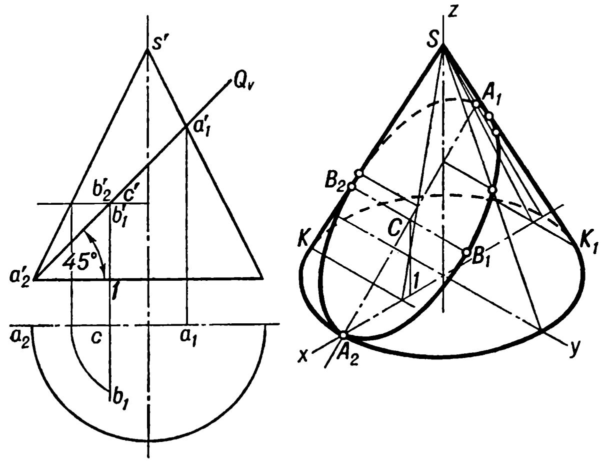


Fig. 475

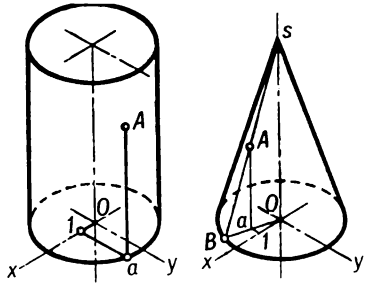


Fig. 476

In the first place we have to find the reference points marked in the drawings by A_1, A_2, B_1, and B_2, as well as those to be obtained on the contour lines of the isometric projection. The semiminor axis of the ellipse obtained in the section equal to cb_1 preserves its length in the isometric projection ($cb_1 = CB_1$). But its sense of the minor axis of an isometric ellipse the line segment B_1B_2 preserves only with the plane Q, i.e. when the angle of inclination of this plane indicated in the drawing is equal to 45°.

Indeed, in this case the segment B_1B_2, being parallel to the y-axis, remains in the isometric projection perpendicular to A_1A_2; hence, the segments A_1A_2 and B_1B_2 retain their sense as the axes of an ellipse. But if a plane is inclined at a different angle, as it is shown on the cylinder for the plane P, the line segments A_1A_2 and B_1B_2 in isometry are no longer the axes of an ellipse, they are only its conjugate diameters.

3. Constructing coordinate segments for a point specified on the surface of a cylinder and a cone of revolution in axonometry. Figure 476 gives a

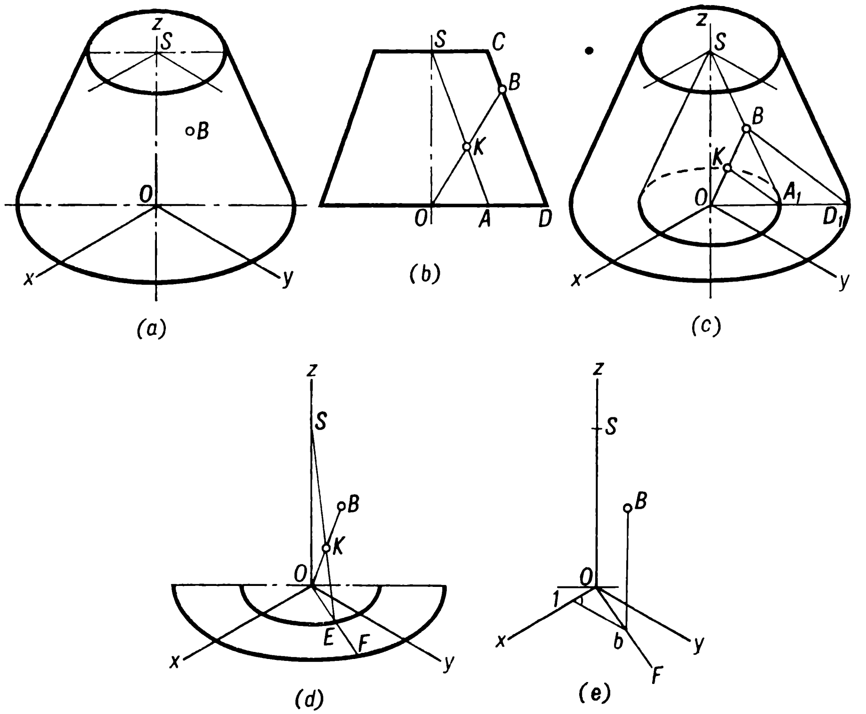


Fig. 477

couple of examples for a cylinder and a cone in isometry. In all the cases under study the origin is taken at the centre of the base (point O).

Through the point A given on the cylinder a straight line is drawn parallel to the z-axis, and from the secondary projection a a straight line is constructed parallel to the y-axis to intersect the x-axis. The line segments $O1$, $1a$, and aA make it possible to determine the coordinates of the point A in the given system of the coordinate axes.

Through the point A given on the cone an element is drawn, and then the secondary projection (OB) of this element is constructed. Drawing from the point A a perpendicular to intersect OB, we get the secondary projection of the point A. The further construction is obvious from the drawing.

Figure 477 illustrates the construction of the coordinate segments for a point specified on the surface of a frustum of a cone of revolution in isometry (Fig. 477, *a*). Suppose we have an axial section of the cone by a plane passing through the point B (Fig. 477, *b*). In the trapezoid thus obtained we draw a straight line SA parallel to CD and a straight line BO to intersect SA at point K. We get the following proportion: $OK : KB = OA : AD$ which is preserved in the isometric projection as well. Let us construct a cone with the vertex at point S whose generatrix will be parallel to that of the frustum of a cone (Fig. 477, *c*). The ratio $OA_1 : A_1D_1$ repeats the ratio $OA : AD$ entering the above proportion. Now we can obtain the point K on OB in Fig. 477, *c*. The element drawn through the points S and E determines the point K (Fig. 477, *d*) and the projection OF of the element on which the point B is situated. Hence we get the possibility to obtain the secondary projection b (Fig. 477, *e*) and coordinate segments Bb, $b1$, and $O1$ determining the coordinates z, y, and x.

The above construction is given for the case when it is impossible to complete the frustum to get a full cone. If such a possibility exists, then the construction is accomplished as is shown for the cone represented in Fig. 477, *b*.

4. Examples of the construction of the lines of mutual intersection of cylindrical and conical surfaces of revolution. The lines of intersection are constructed point by point which are found either by their coordinates taken from the orthogonal projections, or by the method of auxiliary cutting planes applied directly in the axonometric projections. The latter is shown in Fig. 478, *a*, *b*, *c*, *d*.

Auxiliary secant planes intersect the given cylinders and cones along generating elements. In Figure 478, *a* the axes of the cylinders are intersecting lines, while in Fig. 478, *b* they are skew lines. While in (*a*) points A and A_1 are determined by means of a cutting plane passing through the axes of both cylinders, in (*b*) we have to take into account the displacement l. In (*c*) the cutting planes pass through the straight line S_1S_2, and their traces on the plane containing the base of the cone with the vertex S_1 pass through the trace of the line S_1S_2 on this plane. In (*d*) the planes pass through the line MN drawn through the vertex of the cone (point S) parallel to the generatrix of the cylinder.

5. Constructing the points of contact of a circle representing the contour of the projection of a sphere with an ellipse depicting the projection of the circle obtained on the sphere when the latter is cut by a plane. Figure 479, *a* demonstrates a sphere cut by three planes: a profile (T), horizontal (Q), and vertical projecting (S). From this drawing an isometric projection is constructed using the reduced distortion factors (Fig. 479, *b*). The ellipse E_1 is constructed in the way shown in Fig. 469, and the ellipse E_2 as in Fig. 465. The projection of the sphere is represented by a circle of radius equal to $1.22R$. This circle contacts the ellipse E_1 at point K, and the ellipse E_2 at point L.

Let us see how the point K is found. It is obtained on the circle representing the contour of the projection of the sphere, i.e. in the plane of isometric projection (P), and at the same time on the ellipse E_1, i.e. in the plane T which cuts the sphere. But if a point belongs simultaneously to two planes, then it belongs to the line of intersection of these planes.

As is known, the plane of isometric projection is equally inclined to the planes V, H, and W. The triangle of traces of this plane is isosceles (see Fig. 457). Taking the plane P to the point O_p, i.e. to the origin of the axes

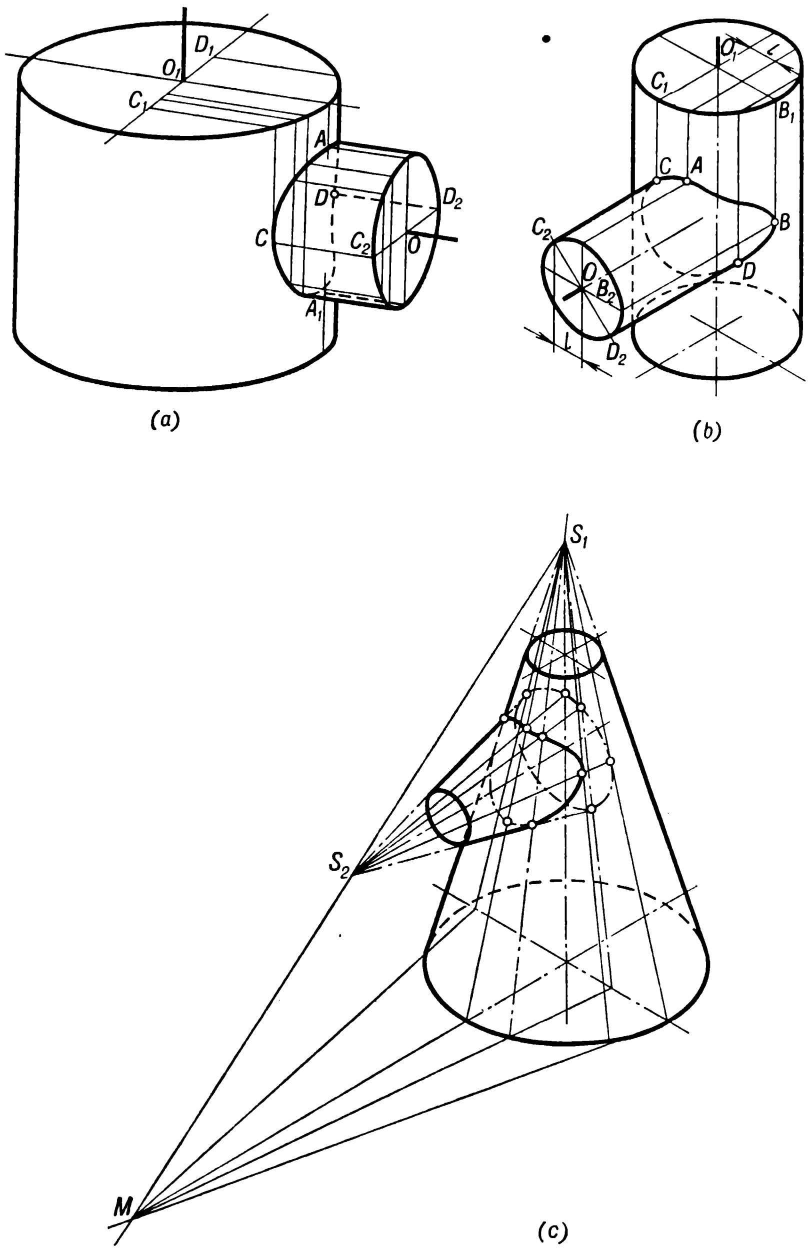


Fig. 478

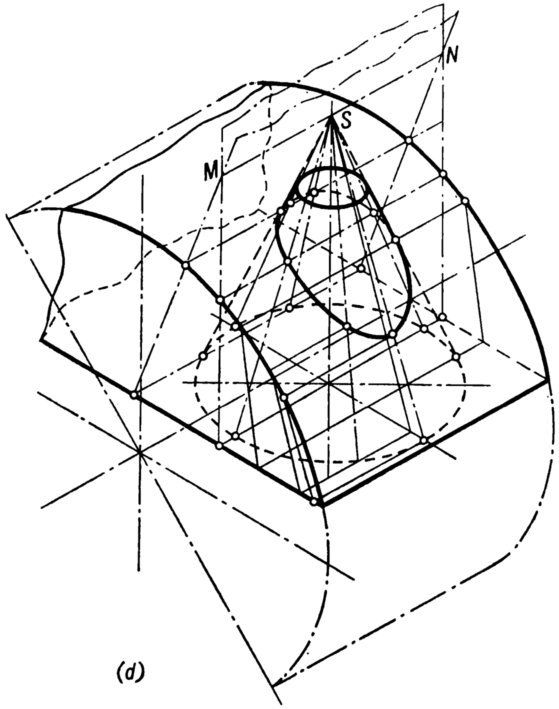


Fig. 478

and centre of the sphere, we get the position of the traces indicated in Fig. 479, *c*.

The plane T will be represented in the same system of axes in traces. as is shown in Fig. 479, *d*. Let us superpose figures (*c*) and (*d*) and construct the line of intersection of the planes P and T (Fig. 479, *e*): the straight line MN passes through the point M of intersection of the horizontal traces parallel to the trace P_w, since T is parallel to W (in this case P_w is perpendicular to O_pX, hence MN is perpendicular to O_px).

Finally, we have to find point K as the point of intersection of the line MN with the circle representing an isometric projection of the sphere (Fig. 479, *f*).

To find the position of point L (see Fig. 479, *b*), we have to represent the vertical projecting plane S (Fig. 479, *g*) in the system of isometric axes, and then to find the line of intersection of the planes P and S (Fig. 479, *c*): this line passes through the point M_1 of intersection of the traces S_h and P_h and through the point N_1 of intersection of the traces S_v and P_v. The required point L is obtained as the intersection of the straight line M_1N_1 with the circle representing an isometric projection of the sphere.

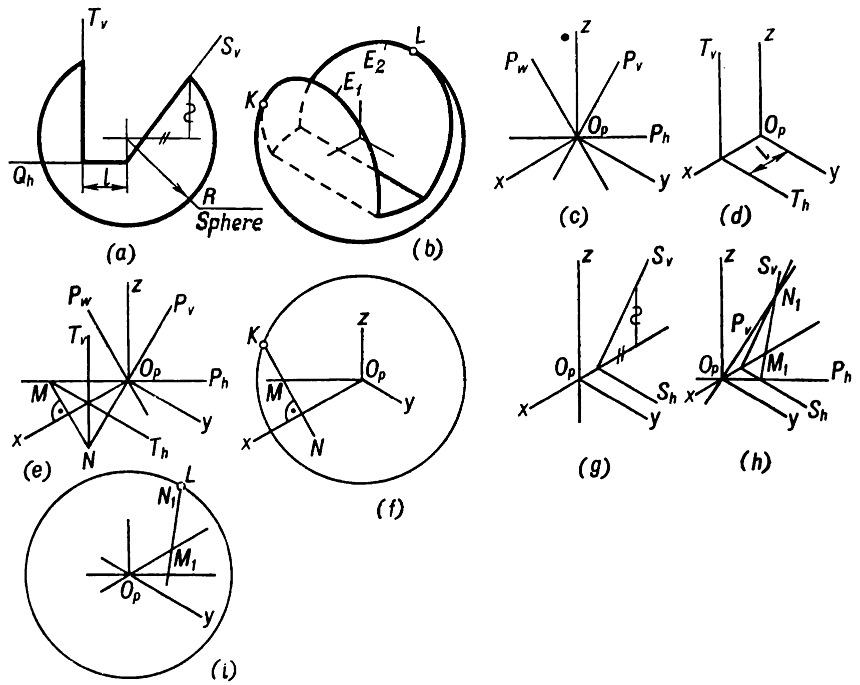


Fig. 479

Sec. 75. Oblique Axonometric Projections

Let us first of all dwell on a frequently used axonometric projection obtained on a plane parallel to the V plane. If the plane of axonometric projections P is parallel to the V plane, then the direction of projecting should not be chosen to be parallel to the W plane, since the projections of the coordinate axes will occupy a position in which the axonometric representation loses its obviousness. The direction of projecting should be chosen so that the projections of the coordinate axes on the plane P are arranged as is shown in Fig. 480. In this case line segments along the x- and z-axes, as well as the angle xO_pz, are projected *without distortion.* Thus, the *distortion factors along the axes O_px and O_pz on the plane P are equal to unity.* As far as the y-axis is concerned, the dist ortionfactor along this axis may have various values, unity included. In the latter case we shall have an *oblique isometric projection.* If the distortion factor along the axis O_py is not equal to unity, then the oblique axonometric projection on the plane P will be *dimetric.*

A line segment OOp parallel to the direction of projecting and line segments Oy and O_1y define the right-angled triangle OyO_1 (the angle OyO_1 is a right one). Indeed, the line segment Oy is perpendicular to the V plane;

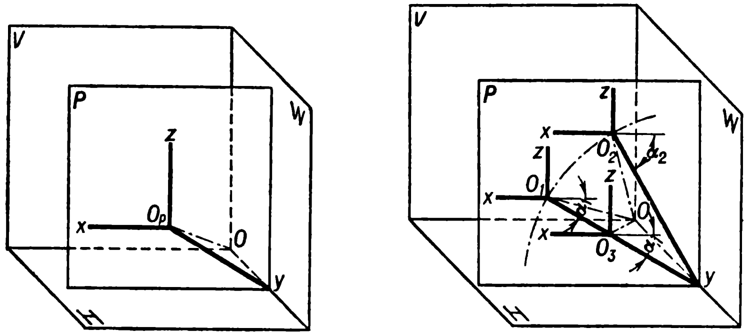


Fig. 480

and since the plane P is parallel to the plane V, the plane P is perpendicular to Oy. Revolving the triangle OyO_1 about the leg Oy, we can obtain various positions of the point O_1 on the plane P. In all its positions the point O_1 is equidistant from the y-axis, hence, the geometric locus of positions of the point O_1 will be a circle of radius yO_1 described from point y as centre. Figure 480 (right) shows two such positions: O_1 and O_2; either of the points O_1 and O_2 serves as the origin of the axes of which the x- and z-axes preserve their directions, and the y-axis changes its direction thus changing the angle between the axonometric axes x and y. This causes a change in the direction of projecting (see the directions of the line segments OO_1 and OO_2). The angle α may be chosen arbitrarily.

On the other hand, if on the plane P the origin is taken at point O_3 on the line segment yO_1, i.e. if we take the direction of projecting to be parallel to the direction of the line segment OO_3, then the magnitude of the angle α_1 remains unchanged, while the ratio $\frac{O_3y}{Oy}$ is not equal to the ratio $\frac{O_1y}{Oy}$. This ratio represents the distortion factor along the y-axis. Consequently, in order to get a most descriptive representation, we may arbitrarily choose both the value of the distortion factor along the y-axis, and the magnitude of the angle α.

The oblique axonometric projection in question, i.e. on a plane parallel to the V plane, is called the vertical projection, as also the *frontal projection*. Use is frequently made of the vertical projection in which the distortion factor along the y-axis is chosen to be equal to 0.5, and the angle α is taken to be equal to 45°. Such projection is sometimes called the *cabinet projection*.

Figure 481 shows a cube represented in cabinet projection. The front face of the cube repeats the projection on the V plane, therefore a circle inscribed in this face will remain a circle in the cabinet projection. From this fact we may draw a conclusion that the cabinet projection, being a very simple and obvious method for representing geometric solids with rectilinear contours, is also convenient for constructing representations when we deal

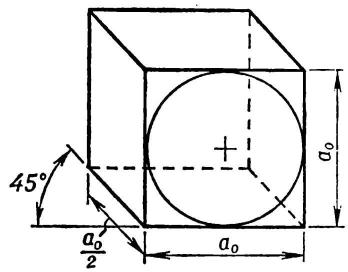


Fig. 481

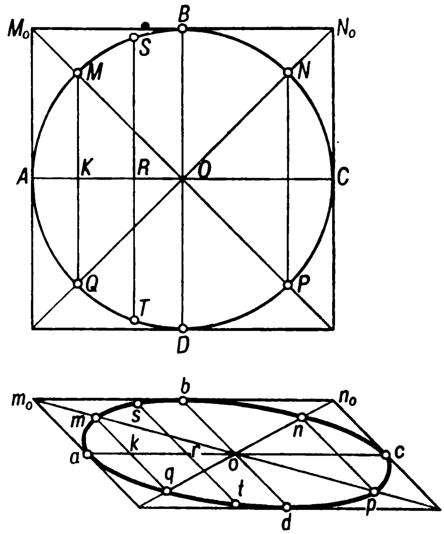


Fig. 482

with circles located in planes parallel to the axonometric plane of projections, i.e. parallel to the V plane.

If it is required to represent in the cabinet projection a circle contained in a plane parallel to the H or W plane, then we inscribe this circle in a square, construct a parallelogram which is a cabinet projection of this square. Then we mark a number of points on the circle and construct their projections. They will be situated on an ellipse into which the circle is projected.

Figure 482 shows the construction of the points belonging to the ellipse which is the projection of a circle located in a plane parallel to the H plane. First of all the circle is inscribed in a square, and the projection of this square is constructed. Here the diameter AC preserves its length and direction (points a and c are obtained); diameter BD perpendicular to AC will occupy a position at an angle of 45° to ac reducing to half its length (points b and d). The chords MQ and NP obtained by drawing the diagonals of the square give another four points (m, q, n, p), where

$$mq = \frac{MQ}{2}, \quad np = \frac{NP}{2}, \quad ok = OK.$$

Furthermore, an arbitrary line segment OR is taken and laid off in the direction oa; through the point r a line segment st is drawn parallel to bd and equal to $ST : 2$. Thus, two more points (s and t) are obtained for the required ellipse. Proceeding in a similar way, we can find a number of points through which the required ellipse passes.

The projection of a circle contained in a plane parallel to the W plane constructed in an analogous manner.

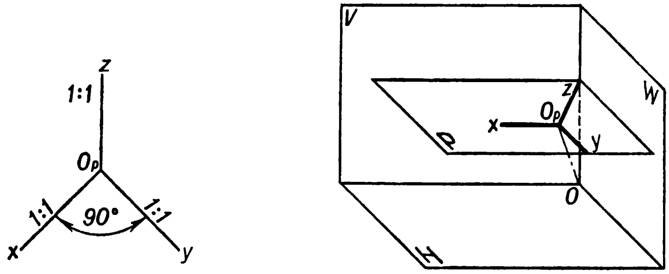


Fig. 483

Let us also consider the case of an oblique axonometric projection when the plane of axonometric projections is parallel to the H plane (Fig. 483). With the plane P so arranged, the angle xO_py is equal to 90°. As to the z-axis obtained on the plane P, the distortion factor along this axis is expressed by the ratio $O_pz : Oz$ (the line segments O_pz and Oz represent the legs of the right-angled triangle OzO_p with a right angle at point z). In the cases when such an oblique axonometric projection is used, the direction of projecting is taken at an angle of 45° to the plane P (or to the H plane). As is obvious, the segment O_pz is equal to Oz, i.e. the distortion factor along the z-axis is equal to unity, and the projection itself turns out to be isometric.

QUESTIONS TO CHAPTER 12

1. What does the method of axonometric projection consist in?
2. What are the distortion factors?
3. What is the secondary projection of a point?
4. How is the transition from rectangular to axonometric coordinates carried out?
5. What is the essence of the basic statement of axonometry (or the basic theorem of axonometry)?
6. In what cases is axonometric projection called: (a) isometric, (b) dimetric, (c) trimetric?
7. What is the difference between an oblique and a rectangular axonometric projection?
8. What line is the contour of the axonometric projection of a sphere: (a) oblique, (b) rectangular?
9. What is the sum of the squares of the distortion factors equal to for a rectangular axonometric projection?
10. What are the distortion factors in a rectangular projection: (a) isometric, (b) dimetric (with the ratio 1 : 0.5 : 1), and what are these factors in the reduced (to unity) form?

11. What is the triangle of traces and what conclusions may be drawn from it in rectangular axonometric projections?

12. How are the axes constructed in rectangular projections: (a) isometric, (b) dimetric (1 : 0.5 : 1)?

13. How do we determine the direction and length of the minor axis of an ellipse representing an isometric or dimetric projection of a circle contained in: (a) an oblique plane, (b) vertical and horizontal projecting planes, (c) a vertical, horizontal, and profile planes?

14. In what cases may a rectangular axonometric projection of a circle turn out to be a line segment or a circle?

15. How do we determine the coordinates of points specified in a rectangular axonometric projection on the surface of: (a) a sphere, (b) a cylinder of revolution, (c) a cone of revolution?

16. What oblique axonometric projection is called: (a) the vertical (or frontal projection), (b) the cabinet projection?

APPENDIX

Sec. 76. Affine Correspondence and Its Application to the Solution of Some Problems

We are going to consider *affine correspondence* of figures located in two intersecting planes or in one plane in the system of parallel projection.

In Figure 484 points A_1 and B_1 contained in the plane T are parallel projected on the plane P in the direction indicated by an arrow. The projecting lines A_1A_2 and B_1B_2 define the projecting plane which intersects the planes T and P along straight lines CB_1 and CB_2 converging on the line MN at point C.

If we take a straight line A_1B_1 in the plane T, then, when extended, the projection of this line on the plane P will meet the line A_1B_1 on the line of intersection of the planes T and P.

Parallel projection of points belonging to the plane T on the plane P establishes correspondence between these planes: to the point A_1 of the plane T there corresponds a point A_2 of the plane P, to the point B_1 a point B_2, and so on. This correspondence possesses the following basic properties:

(1) to every point of one plane there corresponds a unique point of the other plane (one-to-one correspondence);

(2) if on a straight line located in one plane there was established the presence of two points corresponding to the points of a straight line of the other plane, then these lines correspond to each other, and to every point of one of these lines there corresponds a definite point of the other line;

(3) a straight line of one plane intersects with the corresponding straight line of the other plane at a point lying on the line of intersection of the planes*;

---

*If these lines are parallel to the line of intersection of the planes, then the point of intersection of the straight lines is a point at infinity.

(4) the straight line along which the planes intersect corresponds to itself;

(5) if straight lines of one plane are parallel to each other then the corresponding straight lines of the second plane are parallel to each other;

(6) the ratio of two line segments in one plane lying on a single straight line or on parallel lines is equal to the ratio of the corresponding line segments of the other plane.

The considered correspondence between two planes possessing the above listed properties is called *perspective-affine correspondence**. In Figure 484 the points A_2 and B_2 are *affine* to the points A_1 and B_1: straight line A_2B_2 is *affine* to the straight line A_1B_1.

If we take in the plane T a figure and in the plane P consider the points affine to all the points of this figure, then the points on the plane P yield a figure affine to the figure taken in the plane T.

The line MN of intersection of the planes is called the *affine axis.*

The left-hand picture of Fig. 485 shows the same planes in a coincident position: by revolving about the straight line MN, the plane T is brought into coincidence with the plane P. If the direction of revolution is reversed, then the coincident planes will be arranged as is shown in the right-hand picture of the same figure.

If between two planes T and P in space there was established affine correspondence, then, after these planes are brought into coincidence (Fig. 485), this correspondence between their points, straight lines, and figures will be preserved with all the properties of the relationship established in parallel projecting. Indeed, in both cases to a straight line there corresponds a straight line, to a point on one of straight lines there corresponds an affine point on the other, the ratio $\frac{CA_1}{A_1B_1}$ remains equal to the ratio $\frac{CA_2}{A_2B_2}$, and parallelism of the projected lines A_1A_2 and B_1B_2 (Fig. 484) turns into parallelism of the straight lines A_1A_2 and B_1B_2 in Fig. 485 after the planes are brought into coincidence.

Hence, irrespective of whether affine lines are considered in space or in coincident planes, *affine lines intersect on the affine axis and points corresponding to each other lie on parallel lines.*

The direction of the line A_1A_2 is no longer the direction of projection (as in Fig. 484); we shall call it the *affine direction.*

If in the drawing of two coincident planes we are given an affine axis and two affine points, then for every other point of this correspondence we can find an affine point. Suppose (Fig. 486) the line MN is the affine axis, A_1 and A_2 are affine points, and, consequently, A_1A_2 is the affine direction. It is required to find an affine point for the point B_2.

*Perspective-affine correspondence is a special case of affine correspondence of two planes studied in higher geometry. Affinis (Latin) means 'adjacent', 'neighbouring'; affinitas—'relationship', 'property'.

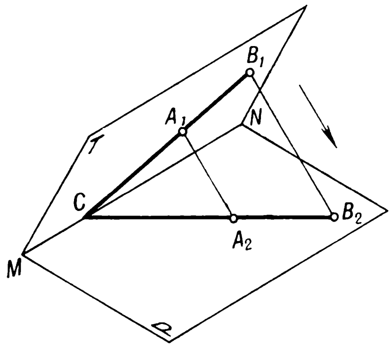


Fig. 484

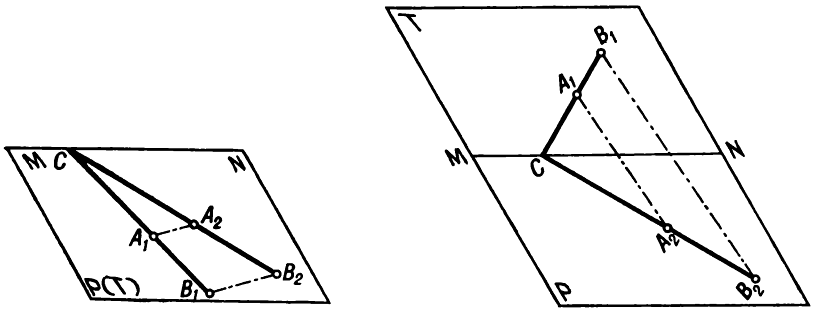


Fig. 485

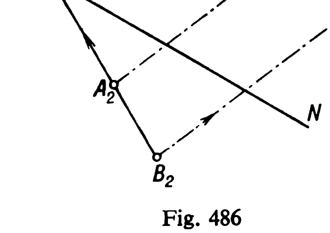


Fig. 486

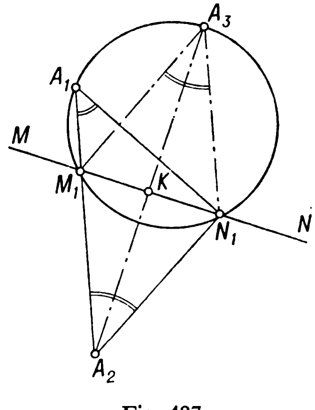


Fig. 487

Proceed as follows: Draw a straight line B_2A_2 to intersect MN; through the points C and A_1 draw a straight line on which the point B_1 affine to B_2 is found by drawing a straight line B_2B_1 parallel to A_2A_1.

Knowing how to construct affine points, we can construct a figure affine to a given one.

If a given figure is a polygon, then the figure affine to it is also a polygon with the same number of sides. To construct this polygon, it is sufficient to find the points affine to the vertices of the given polygon and to join them with rectilinear segments. If we are given a curvilinear figure, then the affine figure is constructed by a number of its points; through the points thus obtained a smooth curve is drawn.

Examining the figure affine to the given one, we note that in the general case the magnitudes of angles are not preserved (see, for instance, Fig. 491: the angles of the quadrilateral $abcd$ are not equal to the corresponding angles of the affine quadrilateral $A_0B_0C_0D_0$).

But with the given affine axis MN (Fig. 487), to the pair of affine points A_1 and A_2 and to the pair of affine straight lines A_1M_1 and A_2M_1 passing through these points, we can construct another pair of affine lines A_1N_1 and A_2N_1 so that the angle $M_1A_1N_1$ will be equal to the angle $M_1A_2N_1$. From the point A_2 a perpendicular is dropped onto the line MN, and a point A_3 is constructed so that $A_2K = KA_3$. Through the points A_1, A_3, and M_1 a circle is drawn which intersects the line MN at point N_1. The further constructions are obvious from the drawing.

In an affine correspondence of two planes specified by the axis and two affine points A_1 and A_2 it is possible to construct two mutually perpendicular directions of one of the planes corresponding to two mutually perpendicular directions of the other plane. *Such directions are called principal in the given affine correspondence.* The relevant construction is given in Fig. 488. The line segment A_1A_2 is bisected at point K, and of this point a perpendicular to A_1A_2 is erected to intersect MN at point C. From the point C as centre a circle is described through the points A_1 and A_2. As a result, two pairs of affine straight lines are obtained: A_1M and A_2M, A_1N and A_2N. MA_1N and MA_2N are right angles.

A figure that is affine (or affine-corresponding) to a circle is, generally, an ellipse. In this case mutually perpendicular diameters of the circle are transformed into conjugate diameters of the ellipse.

Figure 489 represents the affine axis MN and two affine points C_1 and C_2, the former point being the centre of the given circle. The affine direction C_1C_2 is arranged perpendicular to the axis. A figure affine to the circle is constructed, i.e. an ellipse with centre at C_2. The semi-axes A_2C_2 and B_2C_2 of the ellipse are obtained as straight lines affine to two mutually perpendicular radii A_1C_1 and B_1C_1. In this case the right angle A_2C_2K affine to the right angle A_1C_1K is obtained by drawing a straight line A_2C_2 parallel to MN, since C_1A_1 is parallel to MN.

Figure 490 illustrates the construction of the semi-axes A_2C_2 and B_2C_2 of an ellipse affine to a circle with centre at C_1 when the affine direction C_1C_2

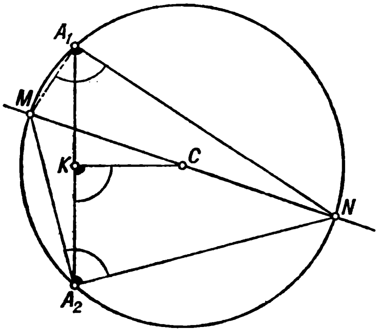


Fig. 488

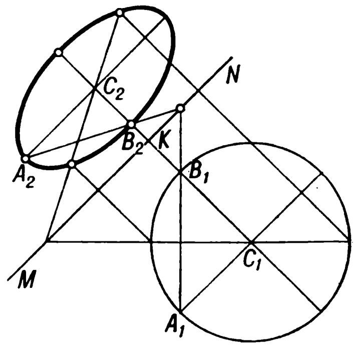


Fig. 489

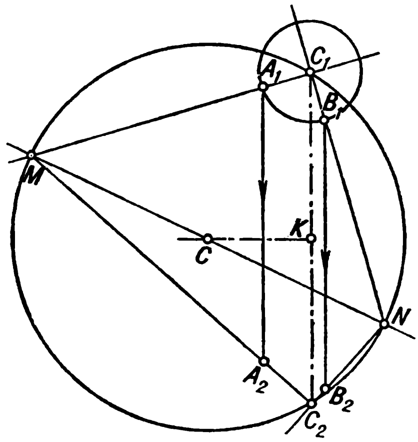


Fig. 490

is not perpendicular to the affine axis. An auxiliary construction (as in Fig. 488) is applied to find the principal directions MC_1 and NC_1, MC_2 and NC_2 which determine the directions of the mutually perpendicular diameters of the circle that are transformed into the axes of the ellipse (Fig. 490 shows only the construction of the semi-axes A_2C_2 and B_2C_2).

If we take an oblique plane in the system of planes V, H, and W, then between the plane P and each of the projection planes there exists the above mentioned affine correspondence, since orthogonal projection is a particular case of parallel projection. Then the traces of the plane P will serve as affine axes: the trace P_h for the planes P and H, the trace P_v for the planes P and V, and the trace P_w for the planes P and W. A straight line located in the plane P and each of its projections intersect on the corresponding traces of the plane, i.e. on the affine axes.

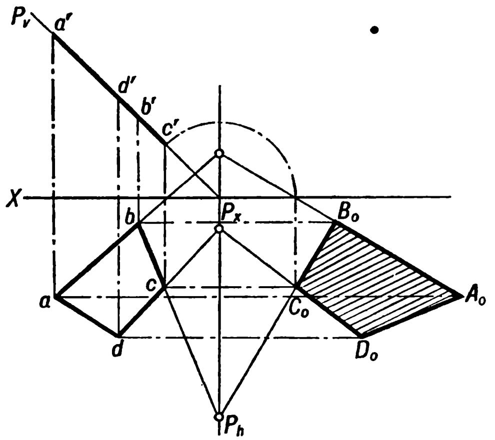


Fig. 491

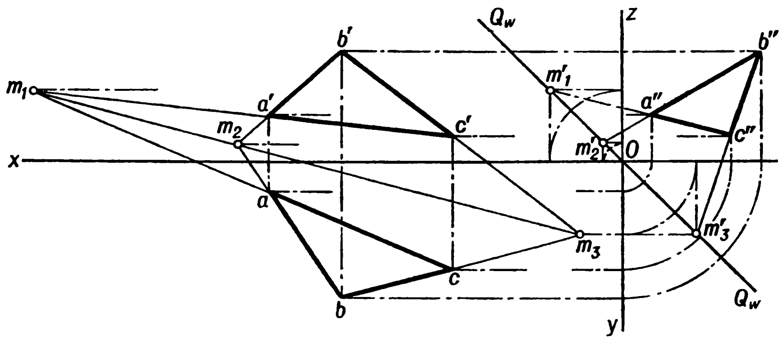


Fig. 492

Figure 491 demonstrates the construction of a quadrilateral $A_0B_0C_0D_0$ (its true size) as a figure affine to the projection *abcd*. The trace P_h of the vertical projecting plane in which the given quadrilateral is situated serves as the affine axes, the affine direction being perpendicular to P_h. First we find in a usual way (applying the coincidence method) point C_0 which is affine to the point c, and then construct the points A_0, B_0, D_0 according to the scheme given in Fig. 486.

Figure 492 shows that *between the horizontal and vertical projections of any plane figure* (in this case of a triangle) *there exists affine correspondence.*

First of all we note that the straight lines joining pairwise the points a and a', b and b', c and c' are parallel. Then we have to establish that any two straight lines corresponding to each other intersect on one and the same straight line. Let us extend the straight lines ab and $a'b'$ until they intersect with each other. The point m_2 represents at the same time the horizontal and vertical projections of the point belonging to the straight line AB in space. The coincidence of the projections testifies to the fact that this point is equidistant from the planes H and V.

The same may be said about the points m_1 and m_3. The fact that the points are equidistant from the planes H and V allows us to conclude that these points, belonging to the plane of the triangle ABC, are at the same time located in the plane bisecting the second and fourth quadrants.

In Figure 492 this plane is represented by the trace Q_w. Since the points under consideration must simultaneously belong to two planes, i.e. to the plane Q and to the plane of the triangle ABC, it is obvious that they must lie on the line of intersection of the plane containing the triangle ABC with the plane Q. Being contained in the plane bisecting the second and fourth quadrants, this line will be represented on the planes H and V by one and the same straight line (the horizontal and vertical projections coincide) and, consequently, the points m_1, m_2 and m_3 are collinear, i.e. they are located on a single line which serves as an affine axis. The projections of any straight line lying in the plane of the triangle ABC intersect on the found affine axis*.

Thus, the projections abc and $a'b'c'$ are affine to each other; the affine direction is perpendicular to the x-axis, and the affine axis is arranged at some angle to the x-axis. If the plane containing the given figure passes through the x-axis, then the affine axis of the horizontal and vertical projections coincides with the x-axis.

For the horizontal and vertical projections of all the figures contained in one and the same plane a common affine axis is obtained. Indeed, this axis represents the coincident horizontal and vertical projections of the line of intersection of some plane with a constant plane Q (Fig. 492).

In Figure 493 affine correspondence is used to construct the horizontal projection of a quadrilateral, provided its vertical projection $a'b'c'd'$ and the horizontal projections of three vertices (points a, b, c) are known.

First of all we find the points m_1 and m_2 thus determining the affine axis. Then we extend the straight line $a'd'$ to intersect the affine axis and join the point m_3 thus obtained to the point a with a straight line.

The required point d is obtained as the intersection of the line am_3 with the line of recall $d'd$. To complete the construction, we join a to d, and c to d with straight lines.

The left-hand drawing of Fig. 494 demonstrates how affine correspondence is applied to find the projections of the point of intersection of the line EF with the plane specified by two parallel lines AB and CD.

The problem is reduced to finding on the straight lines ef and $e'f'$ the points affine to each other in the given affine correspondence. This correspondence is defined by any two affine points (in Fig. 494 points g and g'

*If a straight line lies in the plane containing the triangle ABC and is parallel to the affine axis, then it intersects the latter at infinity, both its projections being parallel to the affine axis.

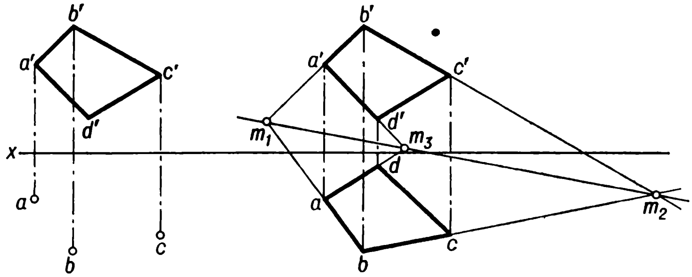


Fig. 493

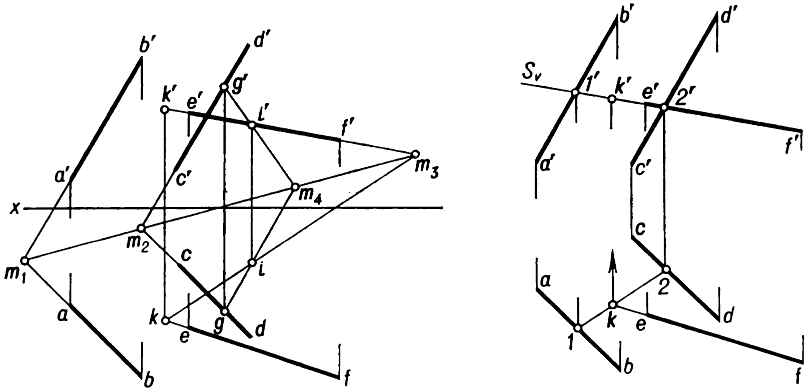


Fig. 494

are taken) and affine axis drawn through the points m_1 and m_2 found as the intersections of the lines *ab* and *a'b'*, *cd* and *c'd'*. If then we construct a straight line affine to *e'f'*, this will mean that in the plane specified by the lines *AB* and *CD* we have drawn a new straight line contained at the same time in one plane with the given line *EF* (a common vertical projection *e'f'*).

A straight line affine to *e'f'* is constructed in the following way: making use of affine points *g* and *g'* and an arbitrarily chosen point *i'* on the line *e'f'*, we construct the point *i* affine to the point *i'*. We then find the point m_3 and through this point and point *i* draw a straight line, thus determining the line affine to the line *e'f'*. The only thing which is left is to mark the point *k* at which the lines im_3 and *ef* intersect each other. This point *k* is the horizontal projection of the required point of intersection.

The right-hand drawing of Fig. 494 shows the solution of the same problem, but using the method set forth in Sec. 25: we pass a plane *S* through the straight line *EF*, construct the line with the projections *1'2'* and *1-2*, along which the plane *S* intersects the given plane, and then obtain the pro-

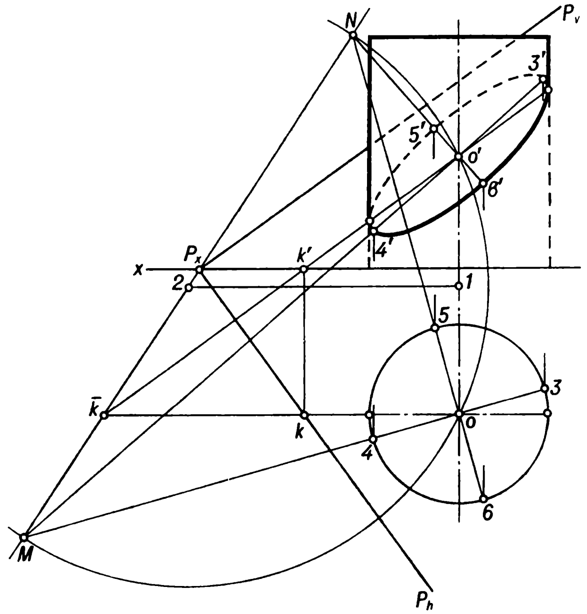


Fig. 495

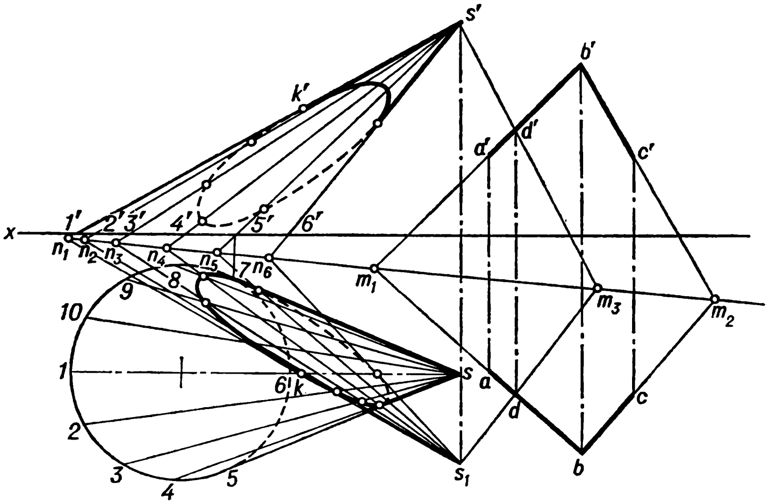


Fig. 496

jection k of the required point; knowing k, we get the projection k'. This construction is simpler as compared with the left-hand drawing.

But in the example given in Fig. 495 the use of affine correspondence makes it possible to construct the axes of the ellipse (which was not done in Figs. 364-366 in Sec. 56) without resorting to transition from its conjugate diameters to axes.

Without explaining how a number of points of the ellipse (representing the vertical projection of the plane section of a cylinder) are found (this was done in Sec. 56), we shall dwell here only on the construction of the axes of the ellipse.

The projections of the section figure—an ellipse and a circle—are affine to each other with the affine direction perpendicular to the x-axis. The affine axis (a straight line MN) is constructed with the aid of the projections $k'o'$ and ko which are affine to each other in the same correspondence and also, say, the trace P_v and x-axis: finding the point $\bar{k}$ and drawing through it and through P_x a straight line, we get the affine axis. Now, using the method shown in Fig. 490, we find mutually perpendicular directions for the vertical projection No' and Mo' and for the horizontal projection No and Mo. Using the points *3* and *4*, we find the vertices *3'* and *4'* of the ellipse on its major axis and making use of the points *5* and *6*, the vertices *5'* and *6'* on the minor axis.

Figure 496 illustrates the intersection of an oblique cone with a plane specified by two intersecting lines AB and BC.

The affine axis, which together with a pair of affine points, say, a and a', defines an affine correspondence, passes through the points m_1 and m_2 of intersection of the projections ab and $a'b'$, bc and $b'c'$. The affine direction is perpendicular to the x-axis.

Since the required section of the cone will be located in the plane defined by the lines AB and BC, the problem is reduced to finding on the projections of the cone a number of pairs of affine points in the given correspondence.

Now we construct the point s_1 affine to the point s' (with the aid of a pair of affine points d and d' and point m_3 on the affine axis).

If we extend the vertical projections of a number of elements of the cone to intersect the affine axis at points n_1, n_2, n_3, and so on, and then join all these points to the point s_1 with straight lines, we shall determine a number of straight lines situated in the given plane. The projections of these lines are affine to each other.

Taking the point of intersection of the horizontal projection of a generating element with the horizontal projection s_1n_1, s_1n_2, etc. which is affine to the vertical projection of this element, we get the horizontal projection of a point belonging to the section figure cut off the cone by the given plane. For instance, the point k is obtained as the intersection of the lines s_1n_1 and $s1$; let us now find the corresponding vertical projection k'. Hence, we have found the point K which lies on one of the elements of the cone and at the same time is located in the given plane.

Finding a number of points in a similar manner, we may construct the ellipses representing the projections of the section figure.

QUESTIONS TO SEC. 76

1. What are the basic properties of the correspondence between two intersecting planes in parallel projecting?
2. How is such correspondence called?
3. What are the affine axis and affine direction?
4. What directions are called principal in a given affine correspondence?
5. What figure is affine to a circle?
6. How do we construct the axes of an ellipse affine to a given circle when the affine direction is not perpendicular to the affine axis?
7. How do we prove that between the vertical and horizontal projections of any plane figure there exists affine correspondence?
8. In what case does the affine axis of the vertical and horizontal projections of a plane figure coincide with the projection axis V/H?

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TO THE READER

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